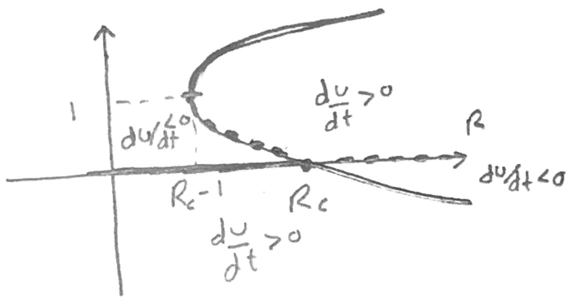
Consider the model equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -u[u^2 - 2u - (R - R_c)]$$

Obtain the steady solutions $u = U$ and draw the bifurcation diagram. Identify the different bifurcations and investigate the stability of the different branches, both by analyzing the sign of $du/dt$ in the different regions of the bifurcation diagram and by performing a "normal mode" analysis with $u = U + \hat{u}e^{st}$.

$$\frac{du}{dt} = 0 \Rightarrow \begin{cases} U = 0 \\ U^2 - 2U - (R - R_c) = 0 \Rightarrow U = 1 \pm \sqrt{1 + (R - R_c)} \end{cases}$$




FOR $U = 0 \rightarrow u = \hat{u}\, e^{st} \rightarrow \quad s = (R - R_c)$ $\quad$ $R > R_c$ UNSTABLE

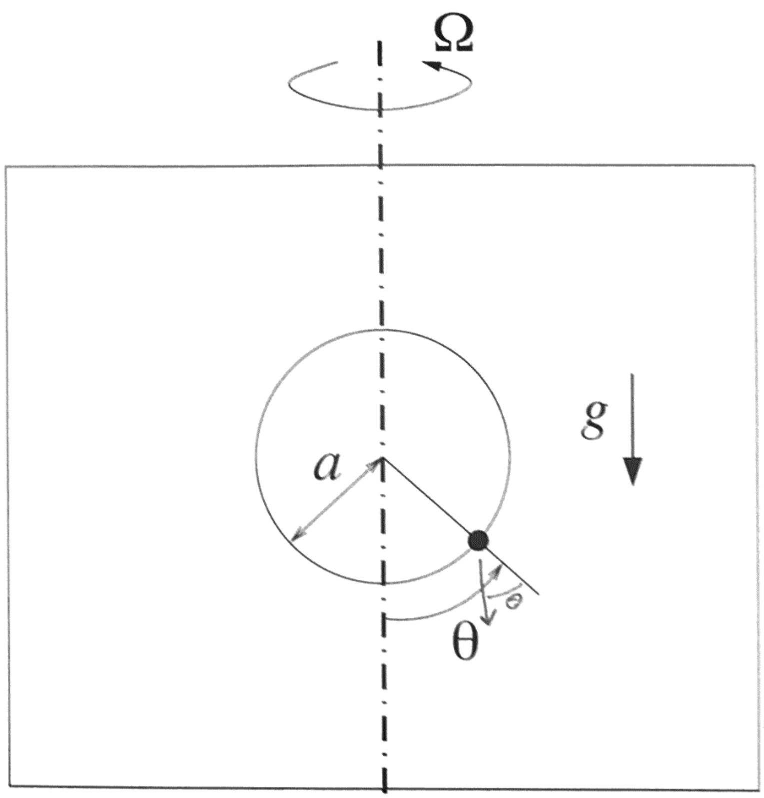
$R < R_c$ ASYMP. STABLE

FOR $u = 1 \pm \sqrt{1 + (R - R_c)} \rightarrow \quad s = -2(U-1)U$ $\begin{cases} U > 1 \rightarrow s < 0 & \text{STABLE} \\ 0 < U < 1 \rightarrow s > 0 & \text{UNSTABLE} \\ U < 0 \rightarrow s < 0 & \text{STABLE} \end{cases}$

(From Drazin, Nonlinear Systems, 1992) A particle is constrained to move around a smooth circular wire of radius $a$ fixed in a vertical plane. Gravity acts on the particle and the plane rotates with constant angular velocity $\Omega$. Show that the particle motion is described by the two first-order differential equations

$$\frac{d\theta}{dt} = \omega \quad \text{and} \quad a\frac{d\omega}{dt} = -g\sin\theta + a\Omega^2\cos\theta\sin\theta,$$

where $\theta$ is the angle between the downward vertical and the radius through the particle, as indicated in the figure. Write the problem in dimensionless form. Find the number of positions of equilibrium $\theta = \Theta$ and sketch the bifurcation diagram in the plane $\theta - \Lambda$ with $\Lambda = a\Omega^2/g$. Analyze the stability of the different branches by performing a "normal mode" analysis with $\theta = \Theta + \hat{\theta}e^{st}$ and $\omega = \hat{\omega}e^{st}$.




IN A REFERENCE FRAME ROTATING WITH THE PLANE THE PROJECTION OF NEWTON'S SECOND LAW IN THE TANGENT DIRECTION LEADS TO

$$m a \frac{d^2\theta}{dt^2} = -mg\sin\theta + m\Omega^2 a\sin\theta\cos\theta \rightarrow \frac{d\theta}{dt} = \omega, \quad a\frac{d\omega}{dt} = -g\sin\theta + a\Omega^2\cos\theta\sin\theta$$

USING $\tau = \Omega t$ AND $\bar{\omega} = \frac{\omega}{\Omega}$ $\Rightarrow$

$$\boxed{\frac{d\theta}{d\tau} = \bar{\omega}, \quad \frac{d\bar{\omega}}{d\tau} = \sin\theta\left(\cos\theta - \frac{1}{\Lambda}\right)}$$

EQUILIBRIUM FOR

$\theta = \Theta = 0$ OR $\Theta = \pi$

$\theta = \Theta = \arccos(\Lambda^{-1})$

$\Theta = \pi \rightarrow \begin{bmatrix} -s & 1 \\ 1+\frac{1}{\Lambda} & -s \end{bmatrix}\begin{bmatrix} \hat{\theta} \\ \hat{\omega} \end{bmatrix} = 0 \rightarrow s = \pm\sqrt{1+\frac{1}{\Lambda}} \rightarrow$ UNSTABLE

$\Theta = 0 \rightarrow s = \pm\sqrt{1-\frac{1}{\Lambda}}$ $\rightarrow \Lambda < 1 \rightarrow$ NEUTRALLY STABLE; $\Lambda > 1 \rightarrow$ UNSTABLE

$\Theta = \arccos(\Lambda^{-1}) \rightarrow \left.\begin{array}{l} s\hat{\theta} = \hat{\omega} \\ s\hat{\omega} = -\left(1-\frac{1}{\Lambda^2}\right)\hat{\theta} \end{array}\right\} \rightarrow s = \pm i\sqrt{1-\frac{1}{\Lambda^2}}$

$\rightarrow$ NEUTRALLY STABLE