Problem 1: Consider the flow of a liquid of density $\rho$ and viscosity $\mu$ confined between two parallel plates separated by a distance $h$. The lower plate is at rest while the upper plate moves parallel to itself with velocity $U_w e_x$. The Navier-Stokes equations can be scaled with $U_w$ and $h$ to give

$$\nabla \cdot \nu = 0$$

$$\frac{\partial \nu}{\partial t} + \nu \cdot \nabla \nu = -\nabla p + \frac{1}{R} \nabla^2 \nu,$$

where $R = \rho U_w h / L$ is the relevant Reynolds number. Consider the linear stability analysis of the steady (Couette) solution $\nu = Zu e_x$ and $P = 0$. Introduce a normal-mode decomposition of the form $\nu = U + \hat{\nu}(z) e^{i(kx+ly-\omega t)}$ and $P = P + \hat{p}(z) e^{i(kx+ly-\omega t)}$, where $k$ and $l$ are non-dimensional wavenumbers and $\omega$ is a non-dimensional complex frequency. Formulate the eigenvalue problem (i.e. differential equations and boundary conditions) that determines the eigenfunctions $\hat{\nu}(z) = [\hat{\nu}_x(z), \hat{\nu}_y(z), \hat{\nu}_z(z)]$ and $\hat{p}(z)$ and the eigenvalue relation $f(\omega, k, l; R) = 0$.

Solution:
We check that the velocity is solenoidal. Indeed, $\nabla \cdot \nu = \nabla \cdot ze_x = 0$. Because the flow is steady and $P$ is a constant, the momentum balance reads $\nu \cdot \nabla \nu = \nabla^2 \nu / R$. The viscous terms are zero because the second spatial derivatives are zero for $\nu = Zu e_x$ and the convective term is equally zero, since $\nu \cdot \nabla \nu = \nabla^2 \nu / 2 = 0$. Upon decomposing our base flow as $\nu = U + \nu'$ and $P = P + p'$ and carrying out a normal–mode decomposition of the form $q' = q e^{i(kx+ly-\omega t)}$, the Navier–Stokes equation reduce to

$$i (k \hat{v}_x + l \hat{v}_y) + \frac{d\hat{v}_z}{dz} = 0$$

$$-i\omega \hat{v}_x + U x i k \hat{v}_x + \hat{v}_x \frac{dU_x}{dz} = -ik \hat{p} + \frac{1}{R} \hat{\nu}^2 \hat{v}_x$$

$$-i\omega \hat{v}_y + U x i k \hat{v}_y = -il \hat{p} + \frac{1}{R} \hat{\nu}^2 \hat{v}_y$$

$$-i\omega \hat{v}_z + U x i k \hat{v}_z = -\frac{dp}{dz} + \frac{1}{R} \hat{\nu}^2 \hat{v}_z,$$

to be integrated with $\hat{v}_x = \hat{v}_y = \hat{v}_z = 0$ at $z = 0, 1$ together with $d\hat{v}_x/dz = 0$ at $z = 0$. We have defined $\nabla = d^2 / dz^2 - (k^2 + l^2)$ above. Eqs.(2)-(4) may be written in compact form as

$$(zk - \omega) i \nu + \hat{v}_x e_x = -i\hat{\nu} \hat{p} + \frac{1}{R} \hat{\nu}^2 \hat{v},$$

where $\nabla = ike_x + ile_y + e_z d/dz$. 
Problem 2: A liquid of density \( \rho \) and viscosity \( \mu \) is enclosed in a square cavity of length \( L \). Consider the planar lid-driven motion induced when one side (e.g., the bottom wall) moves parallel to itself with velocity \( U \). Using the dimensionless vorticity \( \omega(x,y,t) \) and the dimensionless stream function \( \psi(x,y,t) \) together with the Reynolds number \( R = \rho U L / \mu \), show that the problem reduces to that of integrating

\[
-\omega = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2},
\]

with boundary conditions \( \psi = \partial \psi / \partial x = 0 \) at \( x = 0,1 \) for \( 0 \leq y \leq 1 \), \( \psi = \partial \psi / \partial y = 0 \) at \( y = 1 \) for \( 0 \leq x \leq 1 \), and \( \psi = \partial \psi / \partial y - 1 = 0 \) at \( y = 0 \) for \( 0 \leq x \leq 1 \). To analyze the stability of the steady solution \( \omega = \Omega(x,y) \) and \( \psi = \Psi(x,y) \), introduce normal modes of the form \( \omega = \Omega(x,y) + \hat{\omega}(x,y)e^{st} \) and \( \psi = \Psi(x,y) + \hat{\psi}(x,y)e^{st} \). Write the linearized equations with boundary conditions that determine the eigenfunctions \( \hat{\omega}(x,y) \) and \( \hat{\psi}(x,y) \) and accompanying eigenvalues \( s \).

Solution:

Let \( \mathbf{v} \) be the nondimensional velocity field, scaled with \( U \), and \( p \) the nondimensional pressure differences, scaled with \( \rho U^2 \). The nondimensional Navier-Stokes equations for an incompressible flow reduce to

\[
\nabla \cdot \mathbf{v} = 0,
\]

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{R} \nabla^2 \mathbf{v} - \nabla p,
\]

where \( R = \rho U L / \mu \) is the Reynolds number. The boundary conditions are \( \mathbf{v} = 0 \) at the stagnant walls and \( \mathbf{v} = e_x \) at \( y = 0 \) for \( 0 \leq x \leq 1 \). Because the flow is two-dimensional, introduction of the dimensionless stream function \( \psi \) such that \( \partial_y \psi = u_x \) and \( \partial_x \psi = -u_y \) and the \( z \)-component of the dimensionless vorticity \( \omega = \partial_x u_y - \partial_y u_x \) enables the equation to be written in the form

\[
-\omega = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2},
\]

the first stemming from the definition of \( \omega \) and the latter obtained by taking the curl of eq. (8). On introducing the normal mode decomposition described above and neglecting quadratic terms in the perturbations, one finds:

\[
-s \hat{\omega} + \frac{\partial \hat{\psi}}{\partial y} \partial_x \Omega - \frac{\partial \hat{\psi}}{\partial x} \partial_y \Omega + \frac{\partial \Psi}{\partial x} \partial_x \hat{\omega} - \frac{\partial \Psi}{\partial y} \partial_y \hat{\omega} = \frac{1}{R} \left( \frac{\partial^2 \hat{\omega}}{\partial x^2} + \frac{\partial^2 \hat{\omega}}{\partial y^2} \right) \hat{\psi},
\]

which may be integrated with the boundary conditions \( \hat{\psi} = \partial \hat{\psi} / \partial n = 0 \) at all boundaries.

\[\text{It may be useful to recall that}\]

\[
\nabla \wedge (\mathbf{v} \cdot \nabla \mathbf{v}) = \mathbf{v} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{v},
\]

for a solenoidal vector field. Moreover, for a two-dimensional velocity field, \( \omega \cdot \nabla \mathbf{v} = 0 \) (why?). Other useful identities from vector calculus are \( \nabla \cdot (\nabla \wedge \mathbf{a}) = 0 \) for any vector field \( \mathbf{a} \), and \( \nabla \wedge \nabla \phi = 0 \) for any scalar field \( \phi \).