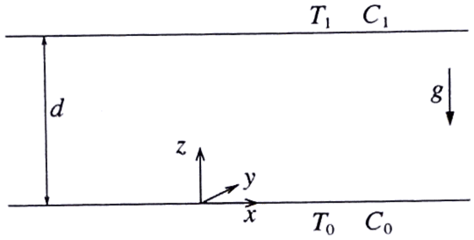
MAE 210C – FLUID MECHANICS III – SPRING 2017

HOMEWORK ASSIGNMENT # 3 (Due at 9:00AM on Wednesday May 31, 2017)

Problem 1 Use the Boussinesq approximation to investigate the motion of a fluid confined between two infinitely long horizontal plates, heated and salted from below, as represented in the figure.



Assume that the density changes are described by the linearized equation of state

$$\frac{\rho-\rho_0}{\rho_0} = -\alpha(T-T_0)+\beta(C-C_0),$$

where C is the concentration of salt, which satisfies the transport equation

$$\frac{\partial C}{\partial t} + \mathbf{v}\cdot\nabla C = D\nabla^2 C,$$

where D is the diffusivity of salt in the fluid. Using d, d^2/D_T, and $\alpha(T_0-T_1)gd^2/\nu$ as scales of length, time, and velocity, show that the problem reduces to the integration of

$$\nabla\cdot\mathbf{v}=0$$

$$\frac{1}{Pr}\frac{\partial \mathbf{v}}{\partial t} + \frac{R}{Pr}\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \theta\mathbf{e}_z - \gamma\phi\mathbf{e}_z + \nabla^2\mathbf{v}$$

$$\frac{\partial\theta}{\partial t} + R\mathbf{v}\cdot\nabla\theta = \nabla^2\theta$$

$$\frac{\partial\phi}{\partial t} + R\mathbf{v}\cdot\nabla\phi = \frac{1}{L}\nabla^2\phi$$

with boundary conditions

$$z=0: \quad \theta=\phi=0 \quad \text{and} \quad \mathbf{v}=0 \text{ (rigid)} \quad \text{or} \quad \frac{\partial v_x}{\partial z}=\frac{\partial v_y}{\partial z}=v_z=0 \text{ (free)}$$

$$z=1: \quad \theta=\phi=-1, \quad \text{and} \quad \mathbf{v}=0 \text{ (rigid)} \quad \text{or} \quad \frac{\partial v_x}{\partial z}=\frac{\partial v_y}{\partial z}=v_z=0 \text{ (free).}$$

In the formulation,

$$\theta=\frac{T-T_0}{T_0-T_1}, \phi=\frac{C-C_0}{C_0-C_1}, R=\frac{\alpha(T_0-T_1)gd^3}{\nu D_T}, \gamma=\frac{\beta(C_0-C_1)}{\alpha(T_0-T_1)}, Pr=\frac{\nu}{D_T}, L=\frac{D_T}{D}.$$

1. Obtain the stagnant base-flow solution $\theta=\Theta(z)$, $\phi=\Phi(z)$, and $p=P(z)$.
2. Write the linearized equations with boundary conditions for the perturbations $\mathbf{v}'$, $\theta'=\theta-\Theta(z)$, $\phi'=\phi-\Phi(z)$, and $p'=p-P(z)$.
3. Introduce normal modes of the form $(v_z', \theta', \phi') = [\hat{v}_z(z), \hat{\theta}(z), \hat{\phi}(x)]e^{st+i(kx+ly)}$ and derive the eigenvalue problem

$$\begin{aligned}
\left(\frac{d^2}{dz^2}-\frac{s}{Pr}-\tilde{k}^2\right)\left(\frac{d^2}{dz^2}-\tilde{k}^2\right)\hat{v}_z &= \tilde{k}^2(\hat{\theta}-\gamma\hat{\phi})\\
\left(\frac{d^2}{dz^2}-s-\tilde{k}^2\right)\hat{\theta} &= -R\hat{v}_z\\
\left(\frac{d^2}{dz^2}-Ls-\tilde{k}^2\right)\hat{\phi} &= -LR\hat{v}_z
\end{aligned}$$

$$z=0,1: \quad \hat{\theta}=\hat{\phi}=\hat{v}_z=0 \quad \text{and} \quad \frac{d\hat{v}_z}{dz}=0 \text{ (rigid)} \quad \text{or} \quad \frac{d^2\hat{v}_z}{dz^2}=0 \text{ (free),}$$

where $\tilde{k}^2=k^2+l^2$.

4. For the case of free boundaries, obtain the eigenvalue relation

$$s^3+(1+Pr+1/L)\lambda^2 s^2+\left[(Pr+1/L+Pr/L)\lambda^4 - Pr\tilde{k}^2(1-\gamma)R/\lambda^2\right]s+\lambda^6 Pr/L+Pr\tilde{k}^2(\gamma-1/L)R=0,$$

where $\lambda^2=(n^2\pi^2+\tilde{k}^2)$.

5. Analyze the different solutions that arise, considering separately the cases $L>1$ and $L<1$. For $L>1$ the bifurcated solutions may be either steady or oscillatory. Investigate in particular the case $L=3$ and $Pr=1$. Identify the different types of behavior in the parametric plane $R-\gamma$ (see the discussion in Huppert & Moore, JFM, 78:821–854 (1976), keeping in mind that their choice of parameters differs from ours).

1) $\Theta = -z,\ \Phi = -z,\ P = -\frac{z^2}{2}(1-\gamma)$

2) $\nabla\cdot\vec{v}' = 0$

$$\frac{1}{Pr}\frac{\partial \vec{v}'}{\partial t} = -\nabla p' + \theta'\vec{e}_z - \gamma\phi'\vec{e}_z + \nabla^2\vec{v}'$$

$$z=0,1:\ \theta'=\phi'=v_z'=0 \begin{cases} \frac{\partial v_x'}{\partial z}=\frac{\partial v_y'}{\partial z}=0 & \text{(FREE)} \\ v_x'=v_y'=0 & \text{(RIGID)}\end{cases}$$

$$\boxed{\begin{aligned}\frac{\partial\theta'}{\partial t} - R v_z' &= \nabla^2\theta' \\ \frac{\partial\phi'}{\partial t} - R v_z' &= \frac{1}{L}\nabla^2\phi'\end{aligned}}$$

$$\nabla\cdot(\ ) \Rightarrow\ 0 = -\nabla^2 p' + \frac{\partial\theta'}{\partial z} - \gamma\frac{\partial\phi'}{\partial z} \Rightarrow\ 0 = -\frac{\partial}{\partial z}\nabla^2 p' + \frac{\partial^2\theta'}{\partial z^2} - \gamma\frac{\partial^2\phi'}{\partial z^2}$$

SUBSTRACTING

$$\frac{1}{Pr}\frac{\partial v_z'}{\partial t} = -\frac{\partial p'}{\partial z} + \theta' - \gamma\phi' + \nabla^2 v_z' \Rightarrow \left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 v_z' = -\frac{\partial}{\partial z}\nabla^2 p' + \nabla^2\theta' - \gamma\nabla^2\phi'$$

$$\boxed{\left(\nabla^2 - \frac{1}{Pr}\frac{\partial}{\partial t}\right)\nabla^2 v_z' = -\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\theta' - \gamma\phi')}$$

3) USING NORMAL MODES:

$$\left.\begin{aligned}&\left(\frac{d^2}{dz^2} - \frac{s}{Pr} - \tilde{k}^2\right)\left(\frac{d^2}{dz^2} - \tilde{k}^2\right)\hat{v}_z = \tilde{k}^2(\hat{\theta} - \gamma\hat{\phi}) \\ &\left(\frac{d^2}{dz^2} - s - \tilde{k}^2\right)\hat{\theta} = -R\hat{v}_z \\ &\left(\frac{d^2}{dz^2} - Ls - \tilde{k}^2\right)\hat{\phi} = -LR\hat{v}_z\end{aligned}\right\}\ z=0,1:\ \hat{\theta}=\hat{\phi}=\hat{v}_z=0 \begin{cases}\frac{d^2\hat{v}_z}{dz^2}=0 & \text{(FREE)} \\ \frac{d\hat{v}_z}{dz}=0 & \text{(RIGID)}\end{cases}$$

4) THE SOLUTION IS $\boxed{\hat{v}_z = \sin(n\pi z) = \frac{[(n\pi)^2 + s + \tilde{k}^2]\hat{\theta}}{R} = \frac{[(n\pi)^2 + Ls + \tilde{k}^2]\hat{\phi}}{LR}}$, LEADING TO

$$s^3 + (1+Pr+1/L)\lambda^2 s^2 + \left[(Pr+\tfrac{1}{L}+Pr/L)\lambda^4 - Pr\tilde{k}^2(1-\gamma)R/\lambda^2\right]s + \lambda^6 Pr/L + Pr\tilde{k}^2(\gamma-\tfrac{1}{L})R = 0$$

5) AT THE MARGIN OF STABILITY $s = i s_I \rightarrow$

$$-(1+Pr+\tfrac{1}{L})\lambda^2 s_I^2 + Pr\tilde{k}^2(\gamma-\tfrac{1}{L})R + \lambda^6 Pr/L$$
$$-s_I^3 + \left[(Pr+\tfrac{1}{L}+Pr/L)\lambda^4 - Pr\tilde{k}^2(1-\gamma)R/\lambda^2\right]s_I = 0$$

IF $s_I = 0 \Rightarrow R_1 = \frac{-\lambda^6}{L\tilde{k}^2(\gamma-\frac{1}{L})} = \frac{(n^2\pi^2+\tilde{k}^2)^3}{\tilde{k}^2(1-L\gamma)} \longrightarrow$ MINIMUM AT $\boxed{\tilde{k} = \frac{n\pi}{\sqrt{2}}}$

IF $s_I \neq 0 \Rightarrow R_2 = \frac{\lambda^6}{Pr\tilde{k}^2}\frac{(Pr+\frac{1}{L})(Pr+1)(\frac{1}{L}+1)}{(1+Pr)(1-\gamma)+\gamma(1-\frac{1}{L})}$

$$\boxed{\begin{aligned}R_1 &= \frac{27}{4}\frac{(n\pi)^4}{(1-L\gamma)} = \frac{27}{4}\frac{\pi^4}{(1-L\gamma)} \\ R_2 &= \frac{27}{4}\frac{\pi^4}{Pr}\frac{(Pr+\frac{1}{L})(Pr+1)(\frac{1}{L}+1)}{(1+Pr)-\gamma(Pr+\frac{1}{L})}\end{aligned}}$$

MINIMUM FOR $n=1$

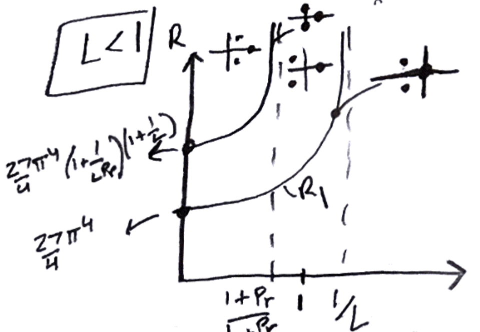
$$R_1 = R_2 \rightarrow \boxed{\begin{aligned}\gamma^* &= \frac{1}{L}\frac{1+Pr}{1+LPr} \\ R^* &= \frac{27}{4}\pi^4\frac{1+LPr}{Pr(L-1)}\end{aligned}}$$

IF $L>1$

$\gamma < \gamma^*$ EXCHANGE OF STABILITIES AT $R=R_1$

$\gamma > \gamma^* \rightarrow$ HOPF BIFURCA[TION AT] $R=R$[2]

$\boxed{L<1}$



$\boxed{L>1}$