THE EQUATION \( p_1 = p_2 \) LEADING TO 3.14 MUST BE REPLACED BY \( p_1 - p_2 = \gamma \left( \frac{d^2 \xi}{dx^2} + \frac{d^2 \xi}{dy^2} \right) \).

So that we get instead

\[
- \frac{d}{\alpha} \left( U_2 \frac{d \phi_1}{dx} + \frac{d \phi_1}{dt} + \frac{d \phi_1}{dy} \right) + \frac{d}{\alpha} \left( U_1 \frac{d \phi_1}{dx} + \frac{d \phi_1}{dt} + \frac{d \phi_1}{dy} \right) \equiv \gamma \left( \frac{d^2 \xi}{dx^2} + \frac{d^2 \xi}{dy^2} \right)
\]

Introducing normal modes and substituting now (3.19) \( A_2 = -(s + ik \cdot U_2) \cdot \frac{1}{k} \) and \( A_1 = (s + ik \cdot U_1) \cdot \frac{1}{k} \), LEADS TO

\[
\frac{d^2}{\alpha} \left[ \frac{\langle x \rangle}{(s + ik \cdot U_2)} \right] - \frac{d^2}{\alpha} \left[ \frac{\langle x \rangle}{(s + ik \cdot U_1)} \right] = \frac{k^2}{\alpha} \gamma
\]

Which can be solved to give

\[
s = -ik \frac{\langle x \rangle U_1 + i U_2}{\langle x \rangle + i U_2} \pm \left[ \frac{k^2 \langle x \rangle \langle x \rangle - U_1 \cdot U_2}{\langle x \rangle + i U_2} \right] \frac{1}{2}
\]

The two modes are neutrally stable provided

\[
k^2 \langle x \rangle \langle x \rangle \langle x \rangle - U_1 \cdot U_2 \langle x \rangle \langle x \rangle + k^2 \gamma
\]

\[
\left( U_1 - U_2 \right) \langle x \rangle \langle x \rangle \langle x \rangle + k^2 \gamma
\]

For a fixed \( k \), the right-hand side is largest for \( \rho = 0 \), \( \frac{\rho}{\langle x \rangle} = k \), so that these are the most unstable conditions. With \( k = \rho \), the least stable wave has a wave number determined by

\[
\max \left[ \frac{k^2 \langle x \rangle \langle x \rangle \langle x \rangle + k^2 \gamma}{\langle x \rangle \langle x \rangle \langle x \rangle} \right] \rightarrow k^2 \gamma = \sqrt{\langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle}
\]

\[
k = \frac{\langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle}{\gamma}
\]

\[
\left( U_1 - U_2 \right)^2 \leq \left( \frac{\langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle}{\gamma} \right) \leq \frac{2 \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle}{\gamma}
\]

At the margin of stability

\[
\Delta U = \left| U_1 - U_2 \right| = \left( \frac{2 \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle}{\gamma} \right)^{1/6} \left( \frac{\langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle}{\gamma} \right)^{1/6} = 6.6 \text{ m/s}
\]

\[
\zeta = \frac{\langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle}{\gamma} = 367 \text{ m}^{-1}, \quad \lambda = \frac{2 \pi k}{\zeta} = 0.0171 \text{ m}
\]

\[
C = \frac{\sqrt{\langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle \langle x \rangle}}{\zeta} \quad \Delta U = \left| U_1 + \frac{6.6}{\zeta} \Delta U \right|
\]

\[
U_2 = U_1 + \Delta U
\]

Relative to the water.
Problem 2: Consider the inviscid parallel flow of two horizontal layers of liquid. The lower layer, of thickness $h$, has density $\rho_1$ and velocity $U_1 e_x$, while the upper layer, of infinite thickness, has density $\rho_2$ and velocity $U_2 e_x$. The lower layer rests upon a horizontal surface, with the interface separating both fluids defined in terms of the vertical distance $z = h + \zeta(x, y, t)$ to that surface.

1. Formulate the problem (equations and boundary conditions) for the two velocity potentials $\phi_1(x, y, z, t)$ and $\phi_2(x, y, z, t)$ and the perturbed interface $\zeta(x, y, t)$.

2. Obtain the base solution for the unperturbed flow $\Phi_1$ and $\Phi_2$.

3. Introduce normal modes proportional to $e^{st+i(kx+ly)}$ and obtain the dispersion relation

$$0 = g\tilde{k}(\rho_1 - \rho_2) + (s + ikU_1)^2 \rho_1 \coth(\tilde{kh}) + (s + ikU_2)^2 \rho_2,$$

where $\tilde{k}^2 = k^2 + l^2$. Solve for $s$.

4. Consider the case $\rho_1 = \rho_2$. Obtain the growth rate and the phase velocity. Show that for $\tilde{kh} \to \infty$ one recovers the results developed in class for Kelvin-Helmholtz instabilities.

5. Consider the case $U_1 = U_2$. Investigate the resulting solutions for $\rho_1 < \rho_2$ and $\rho_1 > \rho_2$, including the limiting cases $\tilde{kh} \ll 1$ and $\tilde{kh} \gg 1$.

Solution:

1. The general equations are those for potential flow, $\nabla_i^2 \phi = 0$ where $i = 1$ corresponds to the region below the interface and $i = 2$ to the upper region. The pressure distribution is given by $P_i = \rho_i \left[ C_i(t) - \frac{\partial \phi_i}{\partial t} - (\nabla \phi_i)^2/2 - gz \right]$, where $C_i(t)$ is a constant of integration. The boundary conditions at the interface include the kinematic and dynamic boundary conditions and the requirement that $\nabla \phi_i \to U_i e_x$ as $z \to 0, \infty$, respectively.

2. The base flow is determined by $\Phi_1 = U_1 x$ and $\Phi_2 = U_2 x$, such that $\nabla \phi_i \to U_i e_x$ at $|z| \to \infty$, $i = 1, 2$. The basic pressure distribution is hydrostatic, namely, $P_i = \rho_i \left[ C_i - (\nabla \phi_i)^2/2 - gz \right]$. The boundary conditions reduce in this case to $[d\Phi/dz] = 0$ and $[P] = 0$, where the brackets indicate jump across the interface.

3. The perturbed problem is described in terms of the perturbed state $\phi' = \hat{\phi}(z)e^{st+i(kx+ly)}$ such that $\phi = \Phi + \phi'$ in each region. The perturbed interface is also describe following the normal mode decomposition $\zeta e^{st+i(kx+ly)}$. The perturbed potentials must remain harmonic so $\nabla^2 \phi'_i = 0$, which yields the equation

$$\frac{d\hat{\phi}_i}{dz} - \tilde{k}^2 \hat{\phi}_i = 0, \quad \tilde{k} = k^2 + l^2, \quad (6)$$

to be solved in each region with the appropriate boundary conditions, namely, $d\hat{\phi}_1/dz = 0$ at $z = 0$ (no penetration at the wall) and $d\hat{\phi}_2/dz \to 0$ as $z \to \infty$. Discarding unbounded solutions at $z \to \infty$, one can write the solution as

$$\hat{\phi} = \begin{cases} 
\hat{\phi}_2 = A_2 e^{-\tilde{k}z}, & z > h + \zeta \\
\hat{\phi}_1 = A_1 \cosh \tilde{k}z, & z < h + \zeta 
\end{cases} \quad (7)$$

where $A_i$ are constants of integration. Kinematic and dynamic boundary conditions should be applied now at $z = h + \zeta$. The former provides a relation between $A_i$ and $\zeta$ as follows

$$(s + ikU_2)\zeta = -A_2 \tilde{k} e^{-\tilde{kh}},$$

$$(s + ikU_1)\zeta = A_1 \tilde{k} \sinh \tilde{kh},$$
which, when combined with the dynamic boundary condition,

\[ \rho_1 \left[ (s + ikU_1)^2 \coth \tilde{k}h + g \right] \hat{\zeta} = \rho_2 \left[ -(s + ikU_2)^2 + g \right] \hat{\zeta}, \]  

renders the dispersion relation as follows

\[ 0 = g\tilde{k}(\rho_1 - \rho_2) + (s + ikU_1)^2 \mathcal{R}_1 + (s + ikU_2)^2 \rho_2, \quad \mathcal{R}_1 = \rho_1 \coth \tilde{k}h. \]  

This is a quadratic equation for \( s \), whose solution is therefore given by

\[ s = -ik \frac{U_1 \mathcal{R}_1 + U_2 \rho_2}{\mathcal{R}_1 + \rho_2} \pm \frac{1}{\mathcal{R}_1 + \rho_2} \sqrt{\mathcal{R}_1 \rho_2 (U_1 - U_2)^2 k^2 - (\mathcal{R}_1 + \rho_2)(\rho_1 - \rho_2) g\tilde{k}} \]  

4. If both fluids have equal densities, eq. (10) can be simplified to

\[ s = \frac{iS + \coth \alpha}{1 + \coth \alpha} \pm \frac{1 - S}{1 + \coth \alpha} \coth^{1/2} \alpha, \quad \alpha = \tilde{k}h > 0, \quad S = U_2/U_1. \]  

We observe that, unless \( S = 1 \) (corresponding to a uniform stream), we can always find an eigenvalue with positive real part, provided \( \coth^{1/2} \alpha/(1 + \coth \alpha) \) is a positive function for positive values of \( \alpha \) (except, maybe, at infinity). We conclude that this configuration is unconditionally unstable when shear is present. The limiting case \( \alpha \gg 1 \), for which \( \coth \alpha \to 1 \), leads

\[ \frac{s}{k} = -i \frac{U_1 + U_2}{2} \pm \frac{|U_1 - U_2|}{2}, \]  

which is the classic result for KHI.

5. When both fluids move at the same velocity, \( U_1 = U_2 = U \), the dispersion relation is such that

\[ s = -ikU \pm \sqrt{-\frac{\rho_1 - \rho_2}{(\mathcal{R}_1 + \rho_2)} g\tilde{k}}, \]  

and hence the flow is unstable iff \( \rho_1 < \rho_2 \), with associated perturbations moving at a phase speed \( c = s_i/k = U \). The limit \( \alpha \gg 1 \), for which \( \mathcal{R}_1 \to \rho_1 \) gives us the characteristic dispersion relation for internal waves whereas the limit \( \alpha \ll 1 \), for which \( \mathcal{R}_1 \gg \rho_2 \), reduces the dispersion relation to

\[ s/k = -ikU \pm \tilde{k}\sqrt{g'h}, \quad g' = g(\rho_2/\rho_1 - 1), \]  

where \( g' \) is the reduced gravity, provided \( 1/\coth \alpha \sim \alpha \) as \( \alpha \to 0 \). This is the dispersion relation for internal gravity waves.
4.1 The analysis is identical to that of Rayleigh with
\[ \vec{v} = \left( \frac{\partial S_x}{\partial x}, -1, \frac{1}{2} \frac{\partial^2 S_y}{\partial y^2} \right) \quad \text{and} \quad \vec{p} = \vec{p}_0 - \frac{\vec{z}}{\alpha}. \]

Correspondingly (4.9) is replaced with
\[ \vec{p} = \gamma \left( \frac{\partial^2 S_y}{\partial x^2} + \frac{\partial^2 S_z}{\partial y^2} \right). \]

A singular behavior of \( \vec{p} \) as \( r \to \infty \) is avoided with \( \vec{p} = \vec{p}_0 - \frac{\vec{z}}{\alpha} \), so that \( \vec{v}_r = -\frac{B}{k} \frac{\partial^2 S_y}{\partial x^2} \).

From (4.9) a \( - \frac{Bk}{9} \frac{\partial^2 S_y}{\partial x^2} = 5 \frac{\vec{z}}{\alpha} \), reducing (4.9b) to
\[ k_n(ka) = \frac{\gamma}{\alpha^3} \left( 1 - (ka)^2 - c^2 \right) \frac{\partial^2 S_y}{\partial x^2} \]
\[ = \begin{cases} \frac{5}{2} \frac{\partial^2 S_y}{\partial x^2} & \alpha \ll 1 \, \text{for} \, \alpha > 0 \\ \frac{3}{2} \frac{\partial^2 S_y}{\partial x^2} + \frac{1}{2} \frac{\partial^2 S_z}{\partial y^2} & \alpha > 1 \end{cases} \]

For \( \eta = 0 \),
\[ \xi = \eta \left( \frac{3}{2}(a^2 \eta^2 - 1) \right)^{1/2} \]

4.3
\[ \vec{v} = \vec{v}_0 + \vec{v}_1 : \]
\[ \vec{p} = \vec{p}_0 + \vec{p}_1 = \gamma \left( \frac{\partial^2 S_y}{\partial x^2} + \frac{\partial^2 S_z}{\partial y^2} \right) \]

Basic solution: \( \vec{v}_0 = 0, \vec{p}_0 = 0 \)

Perturbed solution: \( \vec{v} = 0, \vec{p} = \vec{p}_1 \)

\[ \nabla^2 \vec{p} = 0 \]

At \( z = \pm \alpha \) at all times
\[ \frac{\partial \vec{p}}{\partial t} = \gamma \left( \frac{\partial^2 S_y}{\partial x^2} + \frac{\partial^2 S_z}{\partial y^2} \right) \]

With \( \vec{p}_1 \vec{p}(t, e^{\pm ik_x y}) \)

Using the conditions at \( z = \pm \alpha \)
\[ \nabla^2 \vec{p} = 0 \Rightarrow \vec{p} = A e^{i ka} + B e^{-i ka} \]

Nontrivial solution form
\[ S^2 (A e^{i ka} + B e^{-i ka}) = \pm 2k^3 (A e^{i ka} - B e^{-i ka}) \Rightarrow \left( 2 S^2 e^{2 \alpha} \right) \vec{p} = \left( 2 S^2 e^{2 \alpha} \right) \vec{p}_1 \]

\[ S^2 \left( \frac{\alpha^2}{\alpha^2} \right) = -\alpha^2 \frac{e^+ - e^-}{e^+ + e^-} \Rightarrow -\alpha^2 \text{coth} x \]

Negative for all \( \alpha \), so that \( \eta > 0 \) for all wave numbers.

No
\[ S^2 \frac{2}{\alpha^2} = -\alpha^3 \frac{e^+ + e^-}{e^+ - e^-} = -\alpha^3 \cosh x \]