MAE 210C – FLUID MECHANICS III – SPRING 2017

HOMEWORK ASSIGNMENT # 5 (Due at 11:00AM on Friday June 9, 2017)

Problem The two-dimensional motion and stability of a fluid confined between two long horizontal plates, heated and salted from below, was studied in HWK3 following Huppert & Moore JFM, 78:821–854 (1976). It was seen that introduction of normal modes of the form $(v'_z, \theta', \phi') = [\hat{v}_z(z), \hat{\theta}(z), \hat{\phi}(z)]e^{st+i(kx+ly)}$ leads to the problem

$$
\begin{aligned}
\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{s}{Pr} - \tilde{k}^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \tilde{k}^2\right)\hat{v}_z &= \tilde{k}^2(\hat{\theta} - \gamma\hat{\phi}) \\
\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - s - \tilde{k}^2\right)\hat{\theta} &= -R\hat{v}_z \\
\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - Ls - \tilde{k}^2\right)\hat{\phi} &= -LR\hat{v}_z
\end{aligned}
$$

to be integrated with boundary conditions

$$
z = 0, 1: \qquad \hat{\theta} = \hat{\phi} = \hat{v}_z = 0 \quad \text{and} \quad \frac{\mathrm{d}\hat{v}_z}{\mathrm{d}z} = 0 \text{ (rigid)} \quad \text{or} \quad \frac{\mathrm{d}^2\hat{v}_z}{\mathrm{d}z^2} = 0 \text{ (free)},
$$

where $\tilde{k}^2 = k^2 + l^2$ is the total wavenumber. The present homework is concerned with the numerical solution of this problem with free-free boundary conditions. To that end

1. Rearrange the stability equations as an eigenvalue problem for $s = s_r + \mathrm{i}s_i$.
2. For fixed values of $Pr = 1$ and $L = 2$ and the following combinations of R and γ:

   $$
   \begin{aligned}
   R = 1000: &\quad \gamma = 0.15,\ 0.25 \\
   R = 5000: &\quad \gamma = 0.40,\ 0.80,\ 1.10
   \end{aligned}
   $$

   compute the associated eigenvalue spectra for $0 \leq \tilde{k} \leq 4$. Identify the leading eigenvalue(s), i.e., those with maximum real part, and plot their real and imaginary parts as a function of $\tilde{k}$. For each case, determine the wavenumber for which the growth rate s_r is maximum and plot the corresponding spectrum. In view of your results, place each of the five cases on the transition diagram $R(\gamma)$ (see expressions for $R_1(\gamma)$ and $R_2(\gamma)$ in HWK3 solution).
3. Obtain the critical value of γ for $R = 1000$ and $R = 5000$ and indicate the associated wavenumber. Plot the eigenfunctions $[\hat{v}_z(z), \hat{\theta}(z), \hat{\phi}(z)]$ for these two marginal cases.

Note: Please provide a hardcopy of your source code together with your solution.

MAE210C – Homework assignment 5 – solution

1. Rearranging the stability equations as an eigenvalue problem for s yields

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\tilde{k}^2\right)^2\hat{v}_z-\tilde{k}^2\left(\hat{\theta}-\gamma\,\hat{\phi}\right)=s\,\frac{1}{Pr}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\tilde{k}^2\right)\hat{v}_z, \tag{1}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\tilde{k}^2\right)\hat{\theta}+R\,\hat{v}_z=s\,\hat{\theta}, \tag{2}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\tilde{k}^2\right)\hat{\phi}+L\,R\,\hat{v}_z=s\,L\,\hat{\phi}, \tag{3}$$

subject to the boundary conditions

$$\hat{v}_z=\hat{\theta}=\hat{\phi}=\frac{\mathrm{d}^2\hat{v}_z}{\mathrm{d}z^2}=0\text{ at }z=0,1. \tag{4}$$

2. Figure 1 shows the real and imaginary parts of the leading eigenvalue s as a function of $\tilde{k}$, for different values of R and γ. The red crosses indicate the conditions for which s_r is maximum. The eigenvalue spectra corresponding to the latter conditions are shown in figure 2. The transition diagram $R(\gamma)$ is shown in figure 3, indicating in particular the five cases studied before.

3. The critical conditions are $(R=1000,\gamma=0.171)$, and $(R=5000,\gamma=0.939)$. The associated wavenumber is $\pi/\sqrt{2}\simeq 2.22$. The corresponding eigenfunctions $[\hat{v}_z(z),\hat{\theta}(z),\hat{\phi}(z)]$ are plotted in figure 4.

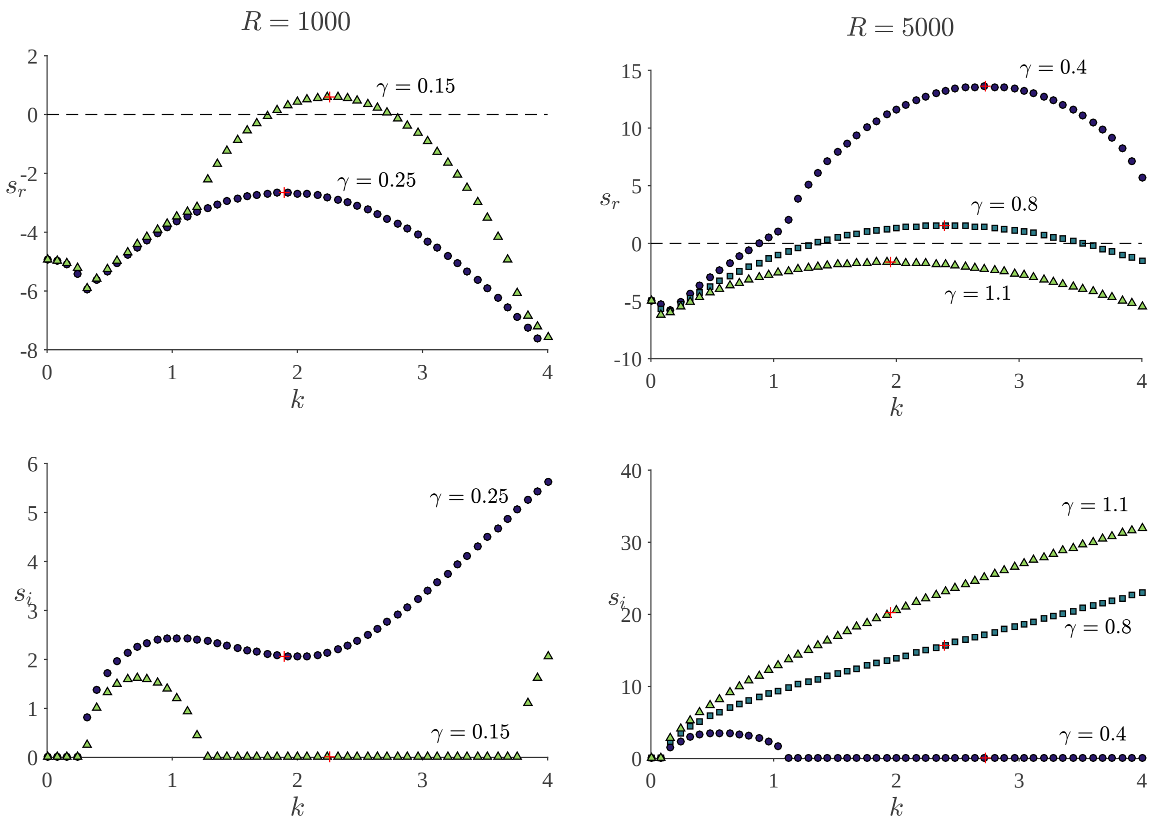

Figure 1: Leading eigenvalue $s = s_r + \mathrm{i}s_i$ as a function of the wavenumber $\tilde{k}$ for different values or R and γ. The red crosses indicate the conditions for which s_r is maximum.

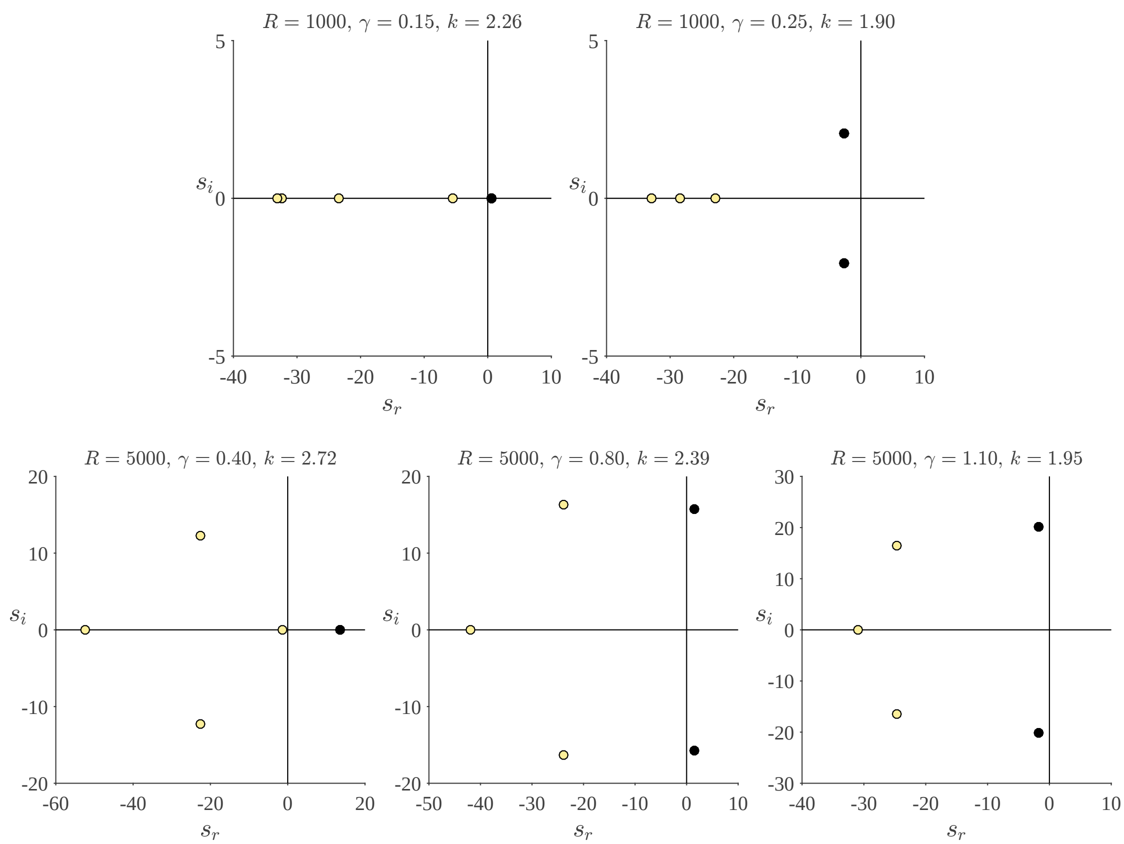

Figure 2: Eigenvalue spectra for different values of R and γ, and values of $\tilde{k}$ for which the real part s_r of the leading eigenvalue is maximum (see red crosses in figure 1).

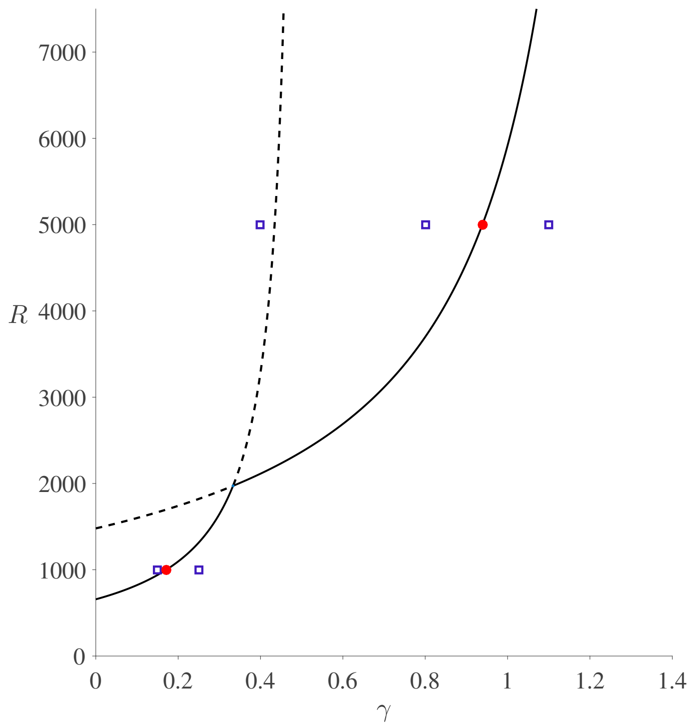

Figure 3: Transition diagram $R(\gamma)$ for *Pr* $= 1$ and $L = 2$. The blue squares indicate the points investigated in figures 1 and 2, whereas the red dots indicate the critical conditions $(R = 1000, \gamma = 0.171)$ and $(R = 5000, \gamma = 0.939)$.

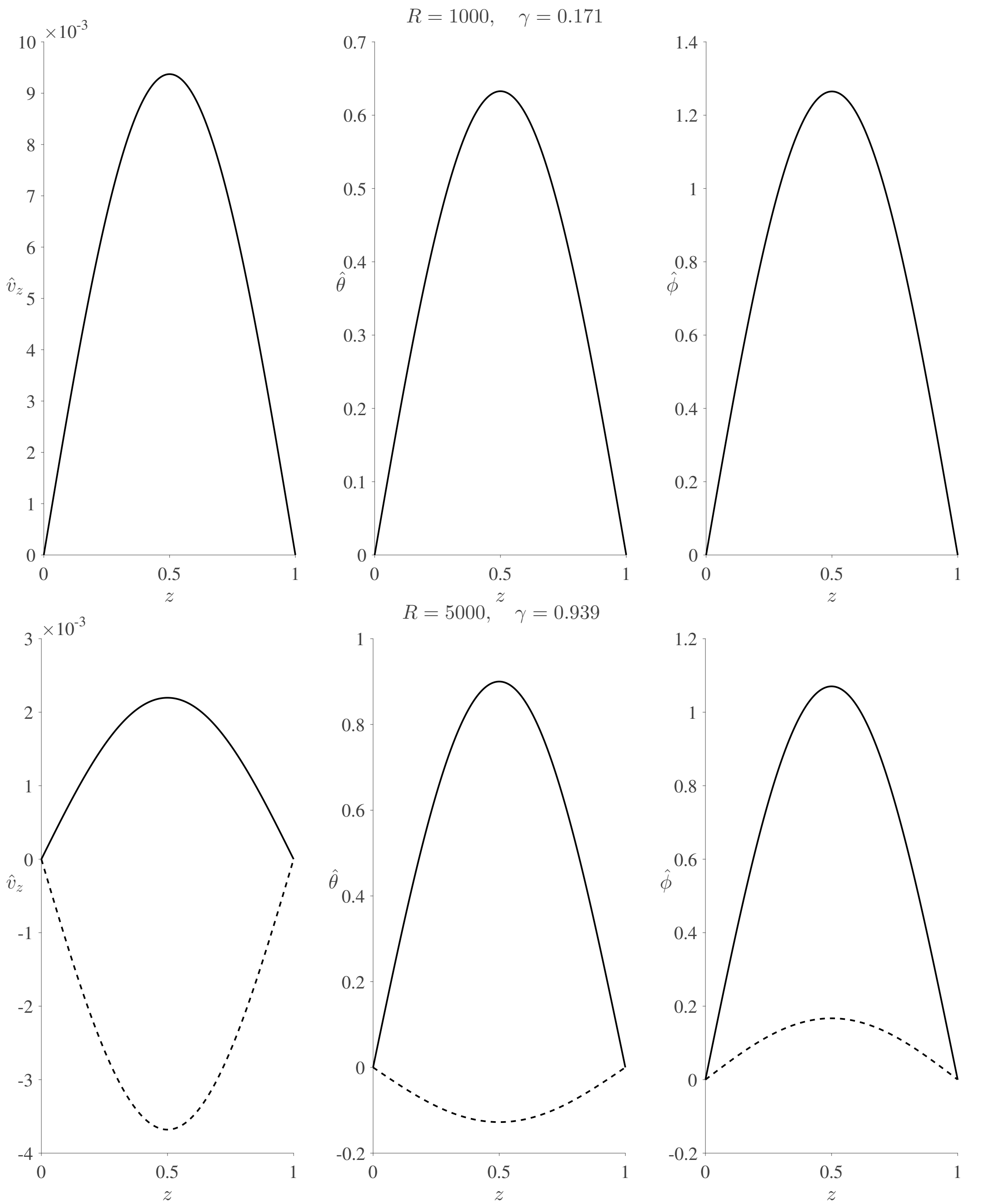

Figure 4: Eigenfunctions $[\hat{v}_z(z), \hat{\theta}(z), \hat{\phi}(z)]$ for two marginally stable cases (dashed lines represent imaginary part).