Problem 1: The linear stability analysis of Rayleigh-Bénard convection simplifies when the perturbations are assumed to be inviscid (of course, you can easily anticipate the resulting criterion for stability). To illustrate this simplified approach, consider the basic solution $\bar{U} = 0$ and $T = T(z)$ corresponding to a fluid layer confined between the two horizontal surfaces $z = z_1$ and $z = z_2 > z_1$.

- Starting from the perturbed form of the inviscid momentum and energy equations
  
  $$\frac{\partial \bar{v}'}{\partial t} = -\nabla (p')/\rho_o + \alpha T'g\bar{e}_z$$
  
  $$\frac{\partial \bar{T}'}{\partial t} + v'z \frac{dT}{dz} = 0$$

  introduce normal modes to show that the growth rate of the inviscid perturbations is determined by integration of

  $$\frac{d^2 \hat{v}_z}{dz^2} - \hat{k}^2 \left( 1 + \frac{\alpha g \frac{dT}{dz}}{s^2} \right) \hat{v}_z = 0, \quad \hat{v}_z(z_1) = \hat{v}_z(z_2) = 0,$$

  all quantities being expressed in dimensional form.

- For a general temperature distribution $T = T(z)$ demonstrate that the flow is stable when $dT/dz > 0$ for $z_1 \leq z \leq z_2$ and unstable when $dT/dz < 0$ for $z_1 \leq z \leq z_2$.

- For the case of constant temperature gradient $dT/dz$, determine the growth rate $s = s_r$ for $dT/dz < 0$ and the angular frequencies $s_i$ for $dT/dz > 0$. 
Poor quality homework will not be accepted. All responses should be written clearly and logically. ALL steps should be clearly shown and well reasoned; lack thereof will result in points being deducted.

**Problem 2:** The axisymmetric viscous stability of a swirling flow of a liquid with kinematic viscosity \( \nu \) and base velocity profile \( \bar{U} = [0, 0, U_\theta(r)] \) admits a general normal-mode formulation in terms of the perturbed variables \([v_x', v_r', v_y', p'/\rho]\) = \( [\hat{v}_x(r), \hat{v}_r(r), \hat{v}_\theta(r), \hat{p}(r)]e^{st+ikx} \), with the continuity and momentum equations expressed in the form

\[
\hat{v}_x = ik^{-1} \left( \frac{d}{dr} + \frac{1}{r} \right) \hat{v}_r \\
\hat{\nu} = \nu \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{r^2 - s^2}{\nu} \right) \left( \frac{d}{dr} + \frac{1}{r} \right) \hat{v}_r \\
\nu \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{r^2 - s^2}{\nu} \right) \hat{v}_\theta = \frac{1}{r} \frac{d}{dr} (rU_\theta) \hat{v}_r
\]

(1)

Use this general framework as a basis to formulate Dean and Görtler stability problems, assuming that the principle of exchange of stabilities holds for both.

**Dean problem:** Investigate the stability of fully developed laminar flow along a cylindrical channel with inner and outer radii \( R_1 \) and \( R_2 \). In the narrow-gap limit, in which the semi-width \( h = (R_2 - R_1)/2 \) is much smaller than the average radius \( R = (R_1 + R_2)/2 \), the basic velocity profile is approximately given by the Poiseuille distribution \( U_\theta/U_{max} = 1 - y^2 \) for \(-1 \leq y \leq 1\), with \( y = (r - R)/h \). Show that the stability margin is determined from the nondimensional eigenvalue problem

\[
\begin{pmatrix}
\frac{d^2}{dy^2} - \bar{k}^2 \\
\frac{d^2}{dy^2} - \bar{k}^2
\end{pmatrix}
\begin{pmatrix}
\hat{v}_r \\
\hat{V}
\end{pmatrix}
= \begin{pmatrix}
\bar{k}^2(1 - y^2)\hat{V} \\
-4D^2y\hat{v}_r
\end{pmatrix}
\]

at \( y = \pm 1 : \hat{v}_r = \frac{d\hat{v}_r}{dy} = \hat{V} = 0 \),

where \( D \) is the so-called Dean number. Show all of the steps involved in the derivation, giving in particular the definition of \( D \).

**Görtler problem:** Consider the stability of a laminar boundary layer of momentum thickness \( \delta \) on a concave wall with curvature radius \( R \gg \delta \). For a basic velocity profile given by the general expression \( U_\theta/U_\infty = F(y) \) in terms of the nondimensional distance to the wall \( y = (R - r)/\delta \), show that the stability margin is determined from the nondimensional eigenvalue problem

\[
\begin{pmatrix}
\frac{d^2}{dy^2} - \bar{k}^2 \\
\frac{d^2}{dy^2} - \bar{k}^2
\end{pmatrix}
\begin{pmatrix}
\hat{v}_y \\
\hat{V}
\end{pmatrix}
= \begin{pmatrix}
-2\bar{k}^2G^2F\hat{V} \\
\frac{4F\hat{v}_y}{\bar{k}}
\end{pmatrix}
\]

at \( y = 0, \infty : \hat{v}_y = \frac{d\hat{v}_y}{dy} = \hat{V} = 0 \),

where \( G \) is the so-called Görtler number and \( \hat{v}_y = -\hat{v}_r \). Show all of the steps involved in the derivation, giving in particular the definition of \( G \).