Problem 1: PIV measurements in a wind tunnel have provided the two velocity profiles shown in the figure below.

Note that both display an inflection point. To investigate their inviscid stability characteristics, we can use Rayleigh’s equation together with the piecewise linear velocity profiles

Profile 1:
\[ U = \begin{cases} 
3(z - 2/3) & \text{for } 2/3 \leq z \leq 1 \\
0 & \text{for } 1/3 \leq z \leq 2/3 \\
3(z - 1/3) & \text{for } 0 \leq z \leq 1/3 
\end{cases} \]

Profile 2:
\[ U = \begin{cases} 
1 & \text{for } 2/3 \leq z \leq 1 \\
6(z - 1/2) & \text{for } 1/3 \leq z \leq 2/3 \\
-1 & \text{for } 0 \leq z \leq 1/3 
\end{cases} \]

which are an approximate representation for an observer moving with the mean velocity (they are indicated by the red broken lines in the figure). Determine in each case the explicit eigenvalue relation \( c(k) \) and discuss the resulting stability characteristics. Could you have anticipated the result with use made of Fjørtoft’s condition?

Problem 2: Consider the wake formed downstream from a cylinder placed perpendicular to a uniform stream. To investigate its inviscid stability characteristics, we can use Rayleigh’s equation together with the approximate piecewise linear representation

Profile 1:
\[ U = \begin{cases} 
1 & \text{for } |z| \geq 1 \\
2(|z| - 1/2) & \text{for } 1/2 \leq |z| \leq 1 \\
0 & \text{for } |z| \leq 1/2 
\end{cases} \]

for the velocity profile. Determine the eigenvalue relation \( c(\alpha) \) in explicit form, considering separately the sinuous and varicose modes. Plot the variation of \( c_r \) and \( c_i \) with \( k \geq 0 \). Discuss your results. Do they (qualitatively) agree with the experimental observations regarding the Von Karman vortex street?
Problem 3: Combustion in nonpremixed systems requires the mixing of the reactants in the shear layers that separate the fuel and air feed streams. Mixing is greatly enhanced when the shear layers are unstable and display transition to turbulent flow. In most applications, the prevailing Mach number is small and the effect of gravity is negligible, so that to investigate the stability response of isothermal fuel-air mixing layers (and planar fuel jets) to inviscid two-dimensional perturbations we can begin by writing the Euler equations in the buoyancy-free form

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0 \\
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} &= -\frac{\nabla p}{\rho} \\
\frac{\partial Y}{\partial t} + \vec{v} \cdot \nabla Y &= 0
\end{align*}
\]

which must be integrated together with the low-Mach-number form of the equation of state

\[
\frac{\rho_{\text{AIR}}}{\rho} = 1 + \left( \frac{\rho_{\text{AIR}}}{\rho_{\text{FUEL}}} - 1 \right) Y,
\]

where \( Y \) is the mass fraction of the fuel.

- Show that one can replace the conservation equations for mass and fuel by the alternative equations

\[
\nabla \cdot \vec{v} = 0 \quad \text{and} \quad \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = 0,
\]

and that the basic solution \( \vec{v} = U(z)\vec{e}_x \), \( \rho = \bar{\rho}(z) \), and \( p = P = \text{constant} \) satisfies the conservation equations.

- Introduce normal modes and derive the eigenvalue problem

\[
(U - c) \left[ \frac{d}{dz} \left( \rho \frac{d\hat{v}_z}{dz} \right) - k^2 \rho \hat{v}_z \right] - \frac{d}{dz} \left( \bar{\rho} \frac{dU}{dz} \right) \hat{v}_z = 0; \quad \hat{v}_z = 0 \text{ at } z = z_1, z_2
\]

- Derive the modified Rayleigh-Fjørtoft’s conditions for these variable-density systems.

The problem admits simplified solutions when the velocity and density profiles are piecewise linear and piecewise uniform, respectively, so that \( \hat{v}_z = A_i e^{kz} + B_i e^{-kz} \) in each subdomain \( i \).

- Use the modified Rayleigh’s equation given above as a starting point to determine the jump conditions that need to be satisfied at the boundaries between subdomains. Interpret your results in physical terms.

- The mathematical framework developed can be used to analyze the temporal inviscid stability of a planar fuel jet discharging into stagnant air, by approximating the velocity and density profiles by the linear functions \( U = 0 \) and \( \rho = 1 \) for \( |z| > 1 \) and \( U = 1 - |z| \) and \( \bar{\rho} = S \) for \( |z| < 1 \). Consider separately the sinuous and varicose modes.

- Discuss the results corresponding to hydrogen (light) jets with \( S < 1 \) and dodecane (heavy) jets with \( S > 1 \).