MAE 210C – FLUID MECHANICS III – SPRING 2018

EXAM # 1 (Due by 11AM on Wednesday May 9, 2018)

ALL responses should be written clearly and logically. ALL steps should be clearly shown and well reasoned; lack thereof will result in points being deducted.

Problem 1: The temperature \( \theta \) of a chemical reactor is determined by the equation

\[
\frac{d\theta}{dt} = -\theta[\theta^2 - 2(D_I - D_E)^{1/2} \theta - (D - D_I)]
\]

where \( D \) is the Damköhler number. We shall see that the known positive constants \( D_E \) and \( D_I > D_E \) characterize the critical conditions at extinction and ignition, respectively.

- Obtain the steady solutions \( \theta = \Theta \) and draw the bifurcation diagram \( \theta - D \).
- Investigate the stability of the different branches by analyzing the sign of \( d\theta/dt \).
- Investigate the stability of the different branches by introducing perturbations of the form \( \theta - \Theta = \hat{\theta}e^{st} \).
- Starting from the solution \( \theta = 0 \) for \( D = 0 \), discuss the evolution of the system when \( D \) oscillates slowly between \( D = 0 \) and \( D = 1.1D_I \) (note that solutions with \( \theta < 0 \) are not physically relevant).

Problem 2: A layer of thickness \( h_2 \) of a liquid with density \( \rho_2 \) rests upon a layer of thickness \( h_1 \) of a liquid with density \( \rho_1 \). The liquids are immiscible and their surface tension is \( \gamma \). The two fluid layers are bounded externally by horizontal walls, as indicated in the figure. Investigate the inviscid stability problem by following the steps listed below.

1. Formulate the problem (equations and boundary conditions) for the two perturbed velocity potentials \( \phi'_1(x, y, z, t) \) and \( \phi'_2(x, y, z, t) \) and the perturbed interface \( \zeta(x, y, t) \).
2. Introduce normal modes proportional to \( e^{st+i(kx+ly)} \) and obtain the eigenvalue relation \( f(s, \tilde{k}; \rho_1, \rho_2, h_1, h_2, \gamma) = 0 \), where \( \tilde{k}^2 = k^2 + l^2 \).
3. Show that the flow is neutrally stable for \( \rho_1 > \rho_2 \) and that the perturbations are internal gravity waves with phase velocity

\[
-\frac{sl}{k} = \pm \left( \frac{(\rho_1 - \rho_2)g/\tilde{k} + \gamma\tilde{k}}{\rho_1 \coth(kh_1) + \rho_2 \coth(kh_2)} \right)^{1/2}
\]

4. Show that for \( \rho_1 < \rho_2 \) the instability modes with sufficiently large wave length are unstable. For the particular case \( h_1 = h_2 = h \), write the growth rate in dimensionless form using \( \kappa = h\tilde{k} \) and \( s^* = s/[(\rho_1 + \rho_2)h^3/\gamma]^{-1/2} \). Obtain the wave number of the most unstable mode and the corresponding growth rate, representing your results as a function of the Bond number \( Bo = (\rho_2 - \rho_1)gh^2/\gamma \).
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**Problem 3:** For flow in porous media, the velocity is approximately described by Darcy’s law \( \mathbf{v} = -(\kappa/\mu)(\nabla p + \rho g\mathbf{e}_z) \), where \( \kappa \) is the intrinsic permeability of the medium. Consider a liquid trapped in a rectangular porous layer of thickness \( d \) and width \( \beta d \) bounded by impermeable surfaces. The lateral surfaces are adiabatic while the lower and upper horizontal surfaces are kept at constant temperatures \( T_0 \) and \( T_1 \), as indicated in the figure.

- The velocity is determined from For strictly planar flow (i.e. \( v_y = \partial \partial y = 0 \)), show that the problem can be formulated in the nondimensional form
  \[
  \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \\
  v_x = -\frac{\partial}{\partial x} + \theta \\
  \frac{\partial^2 v_z}{\partial \tau^2} + R \left( v_z \frac{\partial \theta}{\partial x} + v_z \frac{\partial \theta}{\partial \tau} \right) = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial \tau^2} \\
  \begin{cases} 
  z = 0, 0 \leq x \leq \beta : \quad \theta = v_z = 0 \\
  z = 1, 0 \leq x \leq \beta : \quad \theta + 1 = v_z = 0 \\
  x = 0, 0 \leq z \leq 1 : \quad \partial \theta / \partial x = v_x = 0 \\
  x = \beta, 0 \leq z \leq 1 : \quad \partial \theta / \partial x = v_x = 0
  \end{cases}
  \]
  indicating clearly the selection of dimensionless variables and the definition of the resulting Rayleigh number \( R \).

- Determine the basic solution \( \theta = \Theta \) and \( \pi = \Pi \) corresponding to stagnant flow.

- Introduce perturbed variables \( \tilde{v} = \vec{v}', \theta = \Theta + \Theta', \) and \( \pi = \Pi + \Pi' \) and determine the linearized problem for the perturbations, showing that it can be written in the form
  \[
  \frac{\partial^2 \vec{v}'}{\partial \tau^2} + \frac{\partial^2 \Theta'}{\partial x^2} + \frac{\partial^2 \Theta'}{\partial \tau^2} + R \vec{v}' = 0
  \]
  for which nontrivial solutions exist. Derive an expression for \( \vec{v}' \) and discuss how \( m \) is related to the number of convective cells.

- Solve the eigenvalue problem for \( \{\vec{v}'(z), \Theta(z)\} \) and show that the growth rate is given by
  \[
  s = \frac{R(m\pi/\beta)^2}{(n\pi)^2 + (m\pi/\beta)^2} - \frac{(n\pi)^2 + (m\pi/\beta)^2}{(n\pi)^2 + (m\pi/\beta)^2}
  \]
  where \( m = 1, 2, 3, \ldots \) and \( n = 1, 2, 3, \ldots \).

- Obtain the critical value \( R = R_c(n, m, \beta) \) at the bifurcation. Show that it is minimum for \( n = 1 \). Represent the critical curves \( R_c(1, m, \beta) \) as a function of \( \beta \) for modes with different number of cells \( m = 1, 2, 3, \ldots \).

- From the results, compute the number of cells that would appear at the onset of instability when \( \beta = 0.5 \) and when \( \beta = 2 \). Determine the associated values of \( R_c \). Plot the corresponding streamlines.

\[
\begin{array}{c}
\text{\( T=T_1 \quad v_z=0 \)} \\
\text{dT/\partial x=0}
\end{array}
\quad
\begin{array}{c}
\text{\( T=T_0 \quad v_z=0 \)} \\
\text{dT/\partial x=0}
\end{array}
\]

\[
\begin{array}{c}
\text{\( d \)} \\
\text{\( \beta d \)}
\end{array}
\]