Chapter 4

The Navier-Stokes equations

In many engineering problems, approximate solutions concerning the overall properties of a fluid system can be obtained by application of the conservation equations of mass, momentum and energy written in integral form, given above in (3.10), (3.35) and (3.46), for a conveniently selected control volume. This approach necessitates in general introduction of simplifying assumptions, regarding in particular the spatial distributions of the different variables (e.g., uniform velocity at the open boundaries) and the neglect of terms that are anticipated to give a relatively small contribution to the overall balances. This integral approach is however not suitable when one is interested in computing local properties of the flow (e.g., distributions of velocity $\bar{v}(\bar{x},t)$, density $\rho(\bar{x},t)$, pressure $p(\bar{x},t)$, etc). For that purpose, the conservation equations in integral form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V_f(t)}\rho \mathrm{d}V\right]=0, \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V_f(t)}\rho\bar{v}\mathrm{d}V\right]=-\int_{\Sigma_f(t)}p\bar{n}\mathrm{d}\sigma+\int_{\Sigma_f(t)}\bar{\bar{\tau}}'\cdot\bar{n}\mathrm{d}\sigma+\int_{V_f(t)}\rho\bar{f}_m\mathrm{d}V, \tag{4.2}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V_f(t)}\rho(e+|\bar{v}|^2/2)\mathrm{d}V\right]&=-\int_{\Sigma_f(t)}p\bar{v}\cdot\bar{n}\mathrm{d}\sigma+\int_{\Sigma_f(t)}\bar{v}\cdot\bar{\bar{\tau}}'\cdot\bar{n}\mathrm{d}\sigma\\&+\int_{V_f(t)}\rho\bar{f}_m\cdot\bar{v}\mathrm{d}V-\int_{\Sigma_f(t)}\bar{q}\cdot\bar{n}\mathrm{d}\sigma+\int_{V_f(t)}(Q_\mathrm{c}+Q_\mathrm{r})\mathrm{d}V,\end{aligned} \tag{4.3}$$

need to be transformed with use made of the Gauss formula to express the rate balances occuring locally, providing a set of partial differential equations to be integrated with boundary and initial conditions.

The continuity equation

Using Gauss formula (2.9) it is straightforward to rewrite (4.1) in the form

$$\int_{V_f}\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\bar{v})\right]\mathrm{d}V=0. \tag{4.4}$$

This equation is to be satisfied regardless of the choice of V_f if and only if the identity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0 \tag{4.5}$$

holds at each point of space. This is the continuity (or mass conservation) equation, stating that the sum of the rate of local density variation and the rate of mass loss by convective outflow equals zero. An alternative expression is

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla \cdot \bar{v}, \tag{4.6}$$

indicating that the variation of density following the fluid particle is exclusively due to the rate of volume variation. Note that for an incompressible (constant-density) fluid, the continuity equation reduces to

$$\nabla \cdot \bar{v} = 0, \tag{4.7}$$

a result anticipated previously, whereas for steady gas flow one obtains

$$\nabla \cdot (\rho \bar{v}) = 0, \tag{4.8}$$

that is, the convective rate of mass loss per unit volume is zero.

The momentum equation

The derivation of the momentum equation in differential form follows the procedure used before to obtain (4.5), that is, Gauss formula (2.9) is used to rewrite all of the surface integrals appearing in (4.2) as volume integrals and the resulting integrand is set equal to zero to yield

$$\frac{\partial}{\partial t}(\rho \bar{v}) + \nabla \cdot (\rho \bar{v}\bar{v}) = -\nabla p + \nabla \cdot \bar{\bar{\tau}}' + \rho \bar{f}_m. \tag{4.9}$$

Using now (4.5) and (2.26) provides

$$\rho \frac{\mathrm{D}\bar{v}}{\mathrm{D}t} = \rho \left[\frac{\partial \bar{v}}{\partial t} + \nabla(|\bar{v}|^2/2) - \bar{v} \wedge (\nabla \wedge \bar{v}) \right] = -\nabla p + \nabla \cdot \bar{\bar{\tau}}' + \rho \bar{f}_m, \tag{4.10}$$

which is simply Newton's second law expressed per unit volume of fluid, a result anticipated in (3.25).
For an incompressible fluid with constant viscosity $\nabla \cdot \bar{\bar{\tau}}' = -\mu \nabla \wedge (\nabla \wedge \bar{v})$, so that (4.10) reduces to

$$\frac{\mathrm{D}\bar{v}}{\mathrm{D}t} = -\nabla \left(\frac{p}{\rho} \right) - \nu \nabla \wedge (\nabla \wedge \bar{v}) + \bar{f}_m. \tag{4.11}$$

The viscous force per unit volume can be alternatively expressed as[1] $-\mu \nabla \wedge (\nabla \wedge \bar{v}) = \mu \nabla^2 \bar{v}$, yielding

$$\frac{\mathrm{D}\bar{v}}{\mathrm{D}t} = -\nabla \left(\frac{p}{\rho} \right) + \nu \nabla^2 \bar{v} + \bar{f}_m, \tag{4.12}$$

[1] It is to be noted that in cartesian coordinates each component of the vector $\nabla^2 \bar{v}$ is given simply by the Laplacian of the corresponding velocity component, whereas additional terms appear in cylindrical and spherical coordinates, thereby complicating the computation.

Note that (4.7) and (4.11) suffice to determine the velocity and pressure fields for an incompressible flow with constant viscosity. For such flows, which include those involving water, these two equations are therefore decoupled from the energy equation, which could be used *a posteriori* to determine the temperature field. Also of interest is that, when the mass force is conservative, introduction of $\bar{f}_m = -\nabla U$ enables (4.11) to be written in the form

$$\frac{\mathrm{D}\bar{v}}{\mathrm{D}t} = -\nabla\left(\frac{p+\rho U}{\rho}\right) - \nu\nabla\wedge(\nabla\wedge\bar{v}). \tag{4.13}$$

This last expression suggests that, for the motion of perfect liquids, introduction of the reduced pressure $P = p + \rho U$ (e.g., $P = p + \rho g z$ when $\bar{f}_m = \bar{g}$) may simplify the description significantly. For instance, when the pressure does not enter in the boundary conditions, the solution described with use made of P becomes entirely independent of the mass forces.

A well-known result arises for steady incompressible flows with conservative mass forces when the flow conditions are such that the effect of viscosity is negligible, so that the momentum equation reduces to

$$\nabla\left(\frac{p}{\rho} + U + \frac{|\bar{v}|^2}{2}\right) - \bar{v}\wedge(\nabla\wedge\bar{v}) = 0 \tag{4.14}$$

as can be seen by using (4.13) with $\nu = 0$ together with

$$\frac{\mathrm{D}\bar{v}}{\mathrm{D}t} = \nabla(|\bar{v}|^2/2) - \bar{v}\wedge(\nabla\wedge\bar{v}), \tag{4.15}$$

the corresponding steady form of (2.26). The vector equation (4.14) can be projected along streamlines by multiplying at each point in space by the unit vector $\bar{v}/|\bar{v}|$. Since the contribution of the last term is identically zero, the projection reduces to

$$\frac{\partial}{\partial l}\left(\frac{p}{\rho} + U + \frac{|\bar{v}|^2}{2}\right) = 0, \tag{4.16}$$

where $\partial/\partial l = |\bar{v}|^{-1}(\bar{v}\cdot\nabla)$ denotes the derivative along a stream line. It then follows that **for frictionless steady flow of an incompressible fluid** the quantity

$$p + \rho U + \rho|\bar{v}|^2/2 = C_l \tag{4.17}$$

remains constant along any given stream line, with a value C_l that in general is different for different stream lines. Equation (4.17) is the so-called **Bernoulli's theorem**, that provides a useful relationship between the variations of kinetic energy, potential energy and pressure along streamlines. Note that, if the flow is irrotational, the constant C_l is the same for all streamlines, because $\nabla(p/\rho + U + |\bar{v}|^2/2) = 0$, as follows from (4.14).

The energy equation

Internal energy and kinetic energy conservation equations

The same transformation procedure used above in deriving (4.5) and (4.10) can be employed to derive from (4.3)

$$\frac{\partial}{\partial t}[\rho(e+|\bar{v}|^2/2)] + \nabla\cdot[\rho(e+|\bar{v}|^2/2)\bar{v}] = -\nabla\cdot(p\bar{v}) + \nabla\cdot(\bar{v}\cdot\bar{\bar{\tau}}') + \rho\bar{f}_m\cdot\bar{v} - \nabla\cdot\bar{q} + Q_{\mathrm{c}} + Q_{\mathrm{r}}, \tag{4.18}$$

which can be rewritten by virtue of (4.5) in the simplified form

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(e + |\bar{v}|^2/2) = -\nabla \cdot (p\bar{v}) + \nabla \cdot (\bar{v} \cdot \bar{\bar{\tau}}') + \rho \bar{f}_m \cdot \bar{v} - \nabla \cdot \bar{q} + Q_\mathrm{c} + Q_\mathrm{r}. \tag{4.19}$$

If the mass force derives from a steady potential such that $\bar{f}_m = -\nabla U$ with $\partial U/\partial t = 0$, then the above equation admits the alternative writing

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(e + |\bar{v}|^2/2 + U) = -\nabla \cdot (p\bar{v}) + \nabla \cdot (\bar{v} \cdot \bar{\bar{\tau}}') - \nabla \cdot \bar{q} + Q_\mathrm{c} + Q_\mathrm{r}. \tag{4.20}$$

A separate equation for the mechanical (kinetic) energy $|\bar{v}|^2/2$ can be derived by taking the dot product of (4.10) and $\bar{v}$. Since $\bar{v} \cdot [\bar{v} \wedge (\nabla \wedge \bar{v})] = 0$, the product provides the scalar equation

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(|\bar{v}|^2/2) = -\bar{v} \cdot \nabla p + \bar{v} \cdot (\nabla \cdot \bar{\bar{\tau}}') + \rho \bar{v} \cdot \bar{f}_m, \tag{4.21}$$

which is a purely mechanical energy law. As can be seen, the rate of variation of the kinetic energy following the fluid particle equals the work rate of the mass force $\rho \bar{v} \cdot \bar{f}_m$ plus the work rate of the surface forces associated with the translational motion of the fluid particle $\bar{v} \cdot (\nabla \cdot \bar{\bar{\tau}}) = \bar{v} \cdot (-\nabla p + \bar{\bar{\tau}}')$. Subtracting now (4.21) from (4.19) yields

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p\nabla \cdot \bar{v} + \bar{\bar{\tau}}' : \nabla \bar{v} - \nabla \cdot \bar{q} + Q_\mathrm{c} + Q_\mathrm{r}, \tag{4.22}$$

clearly indicating that the rate of variation of the internal energy is associated with heat addition and with the deformation work rate of the surface forces. The term $-p\nabla \cdot \bar{v}$ corresponds to the compression work, which is a reversible contribution, in that when the fluid volume is reduced $(\nabla \cdot \bar{v} < 0)$ the compression work produces an increase in the internal energy; conversely, the internal energy decreases for positive expansion rates $\nabla \cdot \bar{v} > 0$. The term $\phi_v = \bar{\bar{\tau}}' : \nabla \bar{v}$ is the deformation work due to viscous forces. It is non-negative $(\phi_v \geq 0)$[2], and corresponds to the rate of mechanical energy dissipation per unit volume and unit time. Note that both $-p\nabla \cdot \bar{v}$ and $\bar{\bar{\tau}}' : \nabla \bar{v}$ are independent of the orientation, position and motion of the reference frame.

Integral balance equations for mechanical and internal energy

The integral conservation equation for the energy (4.3) is written for the combined contribution $(e + |\bar{v}|^2/2)$. It is interesting to write separate integral equations for the mechanical and internal energy. To that end, one may integrate (4.21) in the fluid volume $V_f(t)$, bearing in mind the equation

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(|\bar{v}|^2/2) = \frac{\partial}{\partial t}(\rho |\bar{v}|^2/2) + \nabla \cdot (\rho \bar{v} |\bar{v}|^2/2) \tag{4.23}$$

[2] From the definition of contraction of two tensors $\bar{\bar{\mathrm{A}}} : \bar{\bar{\mathrm{B}}} = \sum_\mathrm{i} \sum_\mathrm{j} \mathrm{A}_\mathrm{ij} \mathrm{B}_\mathrm{ij}$ it can be shown that

$$\phi_v = 2\mu \bar{\bar{\mathrm{T}}}_\mathrm{d} : \bar{\bar{\mathrm{T}}}_\mathrm{d} + (\mu_\mathrm{B} - \frac{2}{3}\mu)(\nabla \cdot \bar{\mathrm{v}})^2 = \frac{2\mu}{3}[(\gamma_{11} - \gamma_{22})^2 + (\gamma_{11} - \gamma_{33})^2 + (\gamma_{22} - \gamma_{33})^2 + 6(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2)] + \mu_\mathrm{B}(\nabla \cdot \bar{\mathrm{v}})^2,$$

thereby demonstrating that $\phi_v \geq 0$ provided $\mu > 0$ and $\mu_B > 0$.

and also (3.6), to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V_f(t)} \rho|\bar{v}|^2/2\mathrm{d}V\right] = -\int_{\Sigma_f(t)} p\bar{v}\cdot\bar{n}\mathrm{d}\sigma \quad + \quad \int_{\Sigma_f(t)} \bar{v}\cdot\bar{\bar{\tau}}'\cdot\bar{n}\mathrm{d}\sigma + \int_{V_f(t)} \rho\bar{f}_m\cdot\bar{v}\mathrm{d}V$$

$$- \quad \left[\int_{V_f(t)} -p\nabla\cdot\bar{v}\mathrm{d}V + \int_{V_f(t)} \phi_v\mathrm{d}V\right]. \tag{4.24}$$

Similarly, one may use (4.22) to derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V_f(t)} \rho e\mathrm{d}V\right] = -\int_{\Sigma_f(t)} \bar{q}\cdot\bar{n}\mathrm{d}\sigma \quad + \quad \int_{V_f(t)} (Q_\mathrm{c}+Q_\mathrm{r})\mathrm{d}V$$

$$+ \quad \left[-\int_{V_f(t)} p\nabla\cdot\bar{v}\mathrm{d}V + \int_{V_f(t)} \phi_v\mathrm{d}V\right]. \tag{4.25}$$

The way we have written (4.24) and (4.25) is intended to give a clear picture of the balances of work and energy in the flow field. Thus, the work done by mass forces in the interior of the fluid volume as well as that done by surface forces on its bounding surface (the right-hand-side terms in the first line of (4.24)) are devoted to increase the net amount of mechanical energy contained in the fluid volume. On the other hand, the heat added by conduction, radiation and chemical reaction (the right-hand-side terms in the first line of (4.25)) is employed directly to increase the total internal energy. Viscous dissipation as well as compression work, written for clarity in separate lines in (4.24) and (4.25), are the mechanisms transforming mechanical energy into internal energy, with only the latter being reversible.

Enthalpy conservation equation

Equations. (4.5) and (4.10) together with (4.19) are the so-called Navier-Stokes equations governing fluid motion. The energy equation admits alternative forms, that may be more convenient than (4.19) for the analysis of specific problems. In particular, often in the analysis of gas flows it is of interest to use the enthalpy $h = e + p/\rho$ as a replacement for the internal energy e. In the derivation, one must first use the continuity equation (4.5) to give

$$\nabla\cdot(p\bar{v}) = \rho\frac{\mathrm{D}}{\mathrm{D}t}(p/\rho) - \frac{\partial p}{\partial t}. \tag{4.26}$$

Combining now this result with (4.19) yields

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(h+|\bar{v}|^2/2) = \frac{\partial p}{\partial t} + \nabla\cdot(\bar{v}\cdot\bar{\bar{\tau}}') + \rho\bar{f}_m\cdot\bar{v} - \nabla\cdot\bar{q} + Q_\mathrm{c} + Q_\mathrm{r}, \tag{4.27}$$

which can also be written with use made of (4.21) as a separate equation for the enthalpy

$$\rho\frac{\mathrm{D}h}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + \phi_v - \nabla\cdot\bar{q} + Q_\mathrm{c} + Q_\mathrm{r}. \tag{4.28}$$

Note that, when mass forces are conservative and steady, Eq. (4.27) reduces to

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(h+|\bar{v}|^2/2+U) = \frac{\partial p}{\partial t} + \nabla\cdot(\bar{v}\cdot\bar{\bar{\tau}}') - \nabla\cdot\bar{q} + Q_\mathrm{c} + Q_\mathrm{r}, \tag{4.29}$$

indicating that, when heat addition, temporal pressure variations, and viscous forces are all simultaneously negligible, the combination $h + |\bar{v}|^2/2 + U$ remains constant in the evolution of each fluid particle.

Entropy conservation equation

It is also of interest to express the energy equation in terms of the entropy s. Since $T\mathrm{d}s = \mathrm{d}h - \mathrm{d}p/\rho$, it is straightforward to use (4.28) to write

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = \phi_v - \nabla \cdot \bar{q} + Q_\mathrm{c} + Q_\mathrm{r}, \tag{4.30}$$

indicating that the entropy of a fluid particle increases due to viscous dissipation and heat addition, while the rate of compression work $-p\nabla\cdot\bar{v}$ is reversible and therefore does not change the entropy of the fluid particle. Equation (4.30) can be seen as an application of the second principle of thermodynamics, enabling changes of entropy to be quantified for fluids out of equilibrium. One may divide (4.30) by T and integrate the resulting equation in $V_f(t)$ to give

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{V_f(t)} \rho s \mathrm{d}V\right] &= \int_{V_f(t)} (\phi_v/T)\mathrm{d}V - \int_{V_f(t)} [(\bar{q}\cdot\nabla T)/T^2]\mathrm{d}V \\ &- \int_{\Sigma_f(t)} (\bar{q}/T)\cdot\bar{n}\mathrm{d}\sigma + \int_{V_f(t)} (Q_\mathrm{c}+Q_\mathrm{r})/T\mathrm{d}V, \end{aligned} \tag{4.31}$$

for the variation of the total entropy contained in a fluid volume. The term involving $(Q_\mathrm{c} + Q_\mathrm{r})/T$ represents a volumetric source of entropy associated with chemical reaction and radiation. On the other hand, $\bar{q}\cdot\bar{n}/T$ is the entropy flux across the surface due to heat conduction, so that $-\int_{\Sigma_f(t)}(\bar{q}/T)\cdot\bar{n}\mathrm{d}\sigma$ is the corresponding rate of entropy gain. Viscous dissipation gives a nonnegative contribution $\int_{V_f(t)}(\phi_v/T)\mathrm{d}V$. Finally, the term $-\int_{V_f(t)}[(\bar{q}\cdot\nabla T)/T^2]\mathrm{d}V$ is an additional source of entropy that appears due to conduction inside the control volume. This contribution is also nonnegative, as can be seen with use of (3.44).

Mathematical description of fluid flows

It is a good time to summarize now what we have done so far to derive a rigorous mathematical framework for the description of fluid-flow problems. The derivation given above began by introducing the continuum hypothesis, which was instrumental in defining the concepts of density ρ, velocity $\bar{v}$, and internal energy e as continuous functions of the position $\bar{x}$ and time t. Next, by introducing the hypothesis of local thermodynamic equilibrium, the local values of all of the remaining thermodynamic variables (temperature, pressure, enthalpy, etc) were automatically defined in terms of ρ and e for the non-equilibrium states found in fluid mechanics problems. This hypothesis enables, in particular, ρ and e to be replaced by p and T as fundamental thermodynamic variables for the flow-field description. We have also assumed that the viscous stress tensor $\bar{\bar{\tau}}'$ and the heat-flux vector $\bar{q}$ are linear isotropic functions of the rate of strain and the temperature gradient, respectively. In the constitutive equations arising from this assumption (Navier-Poisson and Fourier laws) there appear three transport coefficients μ, μ_B and k that are assumed to be functions of the local thermodynamic state of the fluid. Although some of the hypotheses introduced can be expected to fail under extreme conditions (rarefied flows, nanofluidics applications, etc), the anticipated range of validity of the mathematical formulation derived covers most applications of interest in engineering.

Summary of conservation equations, equations of state and constitutive equations

To solve a given flow problem, the conservation equations of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) = 0, \tag{4.32}$$

momentum

$$\rho \frac{\mathrm{D}\bar{v}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \bar{\bar{\tau}}' + \rho \bar{f}_m, \tag{4.33}$$

and energy

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p\nabla \cdot \bar{v} + \bar{\bar{\tau}}' : \nabla \bar{v} - \nabla \cdot \bar{q} + Q_{\mathrm{c}} + Q_{\mathrm{r}}, \tag{4.34}$$

must be supplemented with the equations of state

$$\rho = \rho(p, T) \qquad \text{and} \qquad e = e(p, T), \tag{4.35}$$

together with the constitutive equations

$$\bar{\bar{\tau}}' = \mu(\nabla \bar{v} + \nabla \bar{v}^{\mathrm{T}}) + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v})\bar{\bar{\mathrm{I}}} \qquad \text{and} \qquad \bar{q} = -k\nabla T \tag{4.36}$$

and associated state functions

$$\mu = \mu(T), \quad \mu_B = \mu_B(T) \quad \text{and} \quad k = k(T). \tag{4.37}$$

In particular, the state equations (4.35) reduce to

$$\rho = \rho_o \qquad \text{and} \qquad e = e_o + cT \tag{4.38}$$

for a perfect liquid and to

$$\rho = \frac{p}{R_g T} \qquad \text{and} \qquad e = e_o + c_v T \tag{4.39}$$

for a perfect gas.

The vector expressions presented earlier in Chapter 2 enable (4.32), (4.33) and (4.34) to be expressed in any system of orthogonal curvilinear coordinates. For a perfect liquid with constant viscosity, Eqs. (4.32), (4.33) and (4.34) admit the simplified form

$$\nabla \cdot \bar{v} = 0 \tag{4.40}$$

$$\rho \frac{\mathrm{D}\bar{v}}{\mathrm{D}t} = -\nabla p - \mu \nabla \wedge (\nabla \wedge \bar{v}) + \rho \bar{f}_m \tag{4.41}$$

$$\rho c \frac{\mathrm{D}T}{\mathrm{D}t} = 2\mu \bar{\bar{\mathrm{T}}}_{\mathrm{d}} : \bar{\bar{\mathrm{T}}}_{\mathrm{d}} + k\nabla^2 T + Q_{\mathrm{c}} + Q_{\mathrm{r}}, \tag{4.42}$$

which are written out below for cartesian, cylindrical and spherical coordinates.

Initial and boundary conditions

Equations (4.32), (4.33) and (4.34) describe all fluid motions (within the range of validity of the various assumptions introduced in their derivation). The difference between two flow problems lies in their initial and boundary conditions, which need to be specified to enable the integration of the Navier-Stokes equations to be performed.

For unsteady fluid motion, at the initial instant we must define the velocity field $\bar{v} = \bar{v}_o(\bar{x})$ together with the thermodynamic state, given for instance in terms of $p = p_o(\bar{x})$ and $T = T_o(\bar{x})$ (or any other pair of independent thermodynamic variables). For an incompressible fluid, the initial velocity field must necessarily satisfy $\nabla \cdot \bar{v}_o = 0$. Because of the presence of the term $\partial\rho/\partial t$ in (4.32), in the case of gas flow there is no such restriction. Clearly, no initial conditions are required for the study of steady or periodic fluid flows.

The Navier-Stokes equations determine the flow evolution *within* the flow field, and must be therefore complemented with appropriate boundary conditions on the boundaries of the flow field. In typical configurations, the fluid may be confined by solid walls, as occurs in wall-bounded flows. The fluid velocity in contact with the wall will be assumed to be equal to the wall velocity, i.e., at $\bar{x} = \bar{x}_p$ we impose $\bar{v} = \bar{v}_p$, which reduces to $\bar{v} = 0$ when the wall is at rest relative to the reference frame used in the description. This so-called **non-slip condition** comes from assuming that the interaction of the molecules near the wall with the wall is equivalent to that occurring between neighboring particles. For the velocity field to be continuous, no velocity discontinuity is permitted between the wall and the adjacent fluid. There exists ample experimental evidence confirming the validity of the non-slip condition, at least for flows in which the mean free path λ is much shorter than the macroscopic length of the flow field L. Note that, when this is not the case, the non-slip condition must be relaxed, along with the condition of local thermodynamic equilibrium, thereby complicating the associated analysis.

We shall also assume that there exists local thermodynamic equilibrium at the wall, so that the fluid temperature in contact with the wall is that of the wall ($T = T_p$ at $\bar{x} = \bar{x}_p$). There must also exist an equilibrium of the heat fluxes at the wall surface, providing an additional equation that reduces to $\bar{q} \cdot \bar{n} = \partial T/\partial n = 0$ at $\bar{x} = \bar{x}_p$ when the wall is thermally insulated (adiabatic wall). In principle, in integrating the conservation equations for the fluid, one could use alternatively the temperature distribution T_p or the heat flux $\partial T/\partial n$ as boundary condition for the temperature at the wall $\bar{x} = \bar{x}_p$. In many problems, however, neither quantity is known a priori, and they have to be determined as part of the solution, which involves the integration of the Navier-Stokes equations for the fluid together with the heat conduction equation in the solid, coupled with the conditions of thermodynamic equilibrium and heat flux balance at the wall surface $\bar{x} = \bar{x}_p$.

The boundary conditions needed when the fluid extends to infinity (i.e., to distances much larger than the characteristic macroscopic size of the flow field L), as occurs in external aerodynamic problems, include specification of the boundary velocity field and its accompanying thermodynamic state. These external boundary conditions can be written in the form $\bar{v} = \bar{v}_\infty(\bar{x}, t)$, $p = p_\infty(\bar{x}, t)$, and $T = T_\infty(\bar{x}, t)$, for $|\bar{x}| \to \infty$. For instance, to study the motion of a uniform stream of velocity U_∞ over a solid body of characteristic size L, we shall impose $\bar{v} = U_\infty \bar{e}_x$, $p = p_\infty$, and $T = T_\infty$ at $|\bar{x}|/L \gg 1$.

As an example, consider the motion a liquid stream that moves with velocity $U_\infty(t)\bar{e}_x$ over a spherical body of radius R. If the density and transport properties of the fluid are constant and the effect of radiation and chemical reaction are negligible, the problem reduces to that of integrating

$$\begin{aligned} \nabla \cdot \bar{v} &= 0 && (4.43) \\ \rho \frac{\partial \bar{v}}{\partial t} + \rho \bar{v} \cdot \nabla \bar{v} &= -\nabla p + \mu \nabla^2 \bar{v} - \rho g \bar{e}_z && (4.44) \\ \rho c \frac{\partial T}{\partial t} + \rho c \bar{v} \cdot \nabla T &= 2\mu \bar{\bar{\mathrm{T}}}_{\mathrm{d}} : \bar{\bar{\mathrm{T}}}_{\mathrm{d}} + k \nabla^2 T. && (4.45) \end{aligned}$$

Boundary conditions must be imposed on the surface of the body

$$|\bar{x}| = R : \bar{v} = T - T_w = 0, \tag{4.46}$$

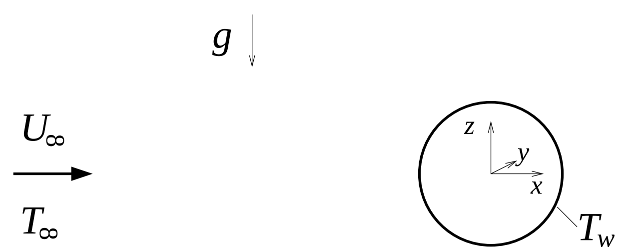
where T_w is the temperature on the surface of the body, different from the value T_∞ of the free stream. Far from the body the velocity should approach that of the free stream whereas the pressure equilibrates the gravity and inertial forces according to

$$|\bar{x}| \gg R : \bar{v} = U_\infty(t)\bar{e}_x, T = T_\infty, p + \rho g z + \rho \frac{\mathrm{d}U_\infty}{\mathrm{d}t} x = constant. \tag{4.47}$$

If the fluid is initially at rest (i.e., $U = 0$ for $t \leq 0$), appropriate initial conditions are

$$t = 0 : \quad \bar{v} = 0, T = T_\infty + \frac{T_w - T_\infty}{|\bar{x}|/R}, \tag{4.48}$$

where the temperature field is that obtained for a fluid at rest through solution of the corresponding reduced energy equation $\nabla^2 T = 0$ with account taken of the spherical symmetry.



Stress-tensor components for a newtonian fluid

The components of the viscous stress tensor can be determined from (3.31) to give

$$\tau'_{ij} = 2\mu\gamma_{ij} + (\mu_B - \frac{2}{3}\mu)\nabla \cdot \bar{v}\delta_{ij},$$

where $\gamma_{ij} = [(\nabla\bar{v})_{ij} + (\nabla\bar{v}^{\mathrm{T}})_{ij}]/2$ is the ij component of the rate-of-strain tensor $\bar{\bar{\mathrm{T}}}_{\mathrm{d}}$ and δ_{ij} represents the Kronecker delta ($\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$). From the expressions derived above in Chapter 2, it can be shown that

$$\gamma_{ii} = \frac{1}{h_i}\frac{\partial v_i}{\partial x_i} + \sum_{k\neq i}\frac{v_k}{h_i h_k}\frac{\partial h_i}{\partial x_k} \quad \text{and} \quad \gamma_{ij} = \frac{h_j}{2h_i}\frac{\partial}{\partial x_i}\left(\frac{v_j}{h_j}\right) + \frac{h_i}{2h_j}\frac{\partial}{\partial x_j}\left(\frac{v_i}{h_i}\right) \quad (\text{if } \ i \neq j).$$

Using this last equation together with

$$\nabla \cdot \bar{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial x_1}(h_2 h_3 v_1) + \frac{\partial}{\partial x_2}(h_1 h_3 v_2) + \frac{\partial}{\partial x_3}(h_1 h_2 v_3)\right],$$

one may write expressions for $\tau'_{ij} = \tau'_{ji}$ in different orthogonal coordinate systems.

Cartesian coordinates ($\nabla \cdot \bar{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$)

$$\tau'_{xx} = 2\mu\frac{\partial v_x}{\partial x} + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v}) \qquad \tau'_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

$$\tau'_{yy} = 2\mu\frac{\partial v_y}{\partial y} + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v}) \qquad \tau'_{yz} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)$$

$$\tau'_{zz} = 2\mu\frac{\partial v_z}{\partial z} + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v}) \qquad \tau'_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

Cylindrical coordinates ($\nabla \cdot \bar{v} = \frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$)

$$\tau'_{rr} = 2\mu\frac{\partial v_r}{\partial r} + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v}) \qquad \tau'_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau'_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v}) \qquad \tau'_{\theta z} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)$$

$$\tau'_{zz} = 2\mu\frac{\partial v_z}{\partial z} + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v}) \qquad \tau'_{zr} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

Spherical coordinates ($\nabla \cdot \bar{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$)

$$\tau'_{rr} = 2\mu\frac{\partial v_r}{\partial r} + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v}) \qquad \tau'_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau'_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v}) \qquad \tau'_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

$$\tau'_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right) + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v}) \qquad \tau'_{\phi r} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

Navier-Stokes equations for a perfect liquid (μ and k constants)

CARTESIAN COORDINATES (x,y,z)

Continuity equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Momentum equation

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho f_{m_x}$$
$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho f_{m_y}$$
$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho f_{m_z}$$

Energy equation

$$\rho c\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = \phi_v + k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + Q_{\mathrm{c}} + Q_{\mathrm{r}}$$

where

$$\phi_v = \mu\left[2\left(\frac{\partial v_x}{\partial x}\right)^2 + 2\left(\frac{\partial v_y}{\partial y}\right)^2 + 2\left(\frac{\partial v_z}{\partial z}\right)^2 + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)^2\right]$$

Navier-Stokes equations for a perfect liquid (μ and k constants)

CYLINDRICAL COORDINATES (r,θ,z)

Continuity equation

$$\frac{1}{r}\frac{\partial r v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Momentum equation

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho f_{m_r}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho f_{m_\theta}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho f_{m_z}$$

Energy equation

$$\rho c\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) = \phi_v + k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right] + Q_{\mathrm{c}} + Q_{\mathrm{r}},$$

where

$$\phi_v = \mu\left\{2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right] + \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]^2 + \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2\right\}$$

Navier-Stokes equations for a perfect liquid (μ and k constants)

SPHERICAL COORDINATES (r,θ,ϕ)

Continuity equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} = 0$$

Momentum equation

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right] + \rho f_{m_r}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right] + \rho f_{m_\theta}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right] + \rho f_{m_\phi}$$

Energy equation

$$\rho c\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial\phi}\right) = \phi_v + k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}\right] + Q_\mathrm{c} + Q_\mathrm{r},$$

where

$$\phi_v = \mu\left\{2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)^2\right] + \left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]^2 + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]^2 + \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]^2\right\}$$