

Chapter 5

Dimensional Analysis

After introducing the subject of dimensional analysis through a simple illustrative example from classical mechanics, we shall present the so-called Π (or Vaschy-Buckingham) theorem, which is next applied to different problems of fluid mechanics. The nondimensionalization of the conservation equations is also introduced as an alternative method to expose the parametric dependence of a fluidmechanical problem. Finally, physical similarity is studied along with its application in scaling experiments.

Dimensions of a physical quantity

We consider the motion of a point particle of mass m that falls from rest from an initial height h under the action of gravity. To compute the collision time t_c at which the particle hits the ground as well as the collision velocity at that instant v_c one may integrate Newton's second law for the motion of the particle $m d^2z/dt^2 = -mg$ with initial conditions $z = h$ and $dz/dt = 0$ at $t = 0$ to give $dz/dt = -gt$ and $z = h - gt^2/2$ for the evolution with time of the particle velocity and of the instantaneous distance to the ground z . Equating the last expression to zero yields $t_c = \sqrt{2h/g}$ for the collision time, which can be used to give $v_c = gt_c = \sqrt{2hg}$ for the collision velocity. The objective of this introductory section is that of showing how the solution can be determined almost completely by a simple reasoning based on dimensional arguments.

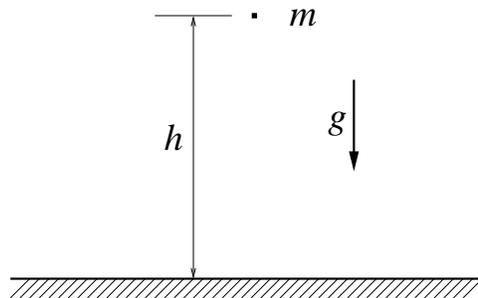


Figure 5.1: The fall of a point mass.

As a first step in solving any given problem in engineering or physics we need to identify the dependences of the unknown quantities, i.e., the governing parameters that influence the solution.

For the problem at hand, one would in principle guess that the values of the collision time and collision velocity are functions of the mass of the body m , the falling distance h , and the gravity g (because that is, after all, the driving mechanism for the body motion), so that we write

$$t_c = t_c(m, h, g) \quad \text{and} \quad v_c = v_c(m, h, g). \quad (5.1)$$

The numbers t_c , v_c , h , m , and g represent the magnitudes of physical quantities. To obtain these numbers we need to define a system of fundamental units of measurement. For instance, one may select the second, the minute, or the hour as the unit of time. The value of t_c is obtained by comparing the collision time with the unit of time, giving a number that depends on the unit of time selected (for instance, $t_c = 180$ seconds would be $t_c = 3$ minutes or $t_c = 0.05$ hours). Similarly, to complete the system of units we define a unit of length (centimeter, or inch, or yard) and a unit of mass (kilogram, or pound, or ton). The comparison of the initial height with the unit length would provide the number h while the comparison of the mass of the body with the unit of mass would give the number m . From the fundamental units of length and time one may construct a unit of velocity, as the velocity of a moving body that travels the unit of length in the unit time. Comparing the impact velocity of the body with the unit of velocity would give the number v_c (e.g., $v_c = 360$ Km/h or $v_c = 100$ m/s). Correspondingly, we shall define the unit of acceleration as the that of a body that increments its velocity by the unit of velocity in the unit of time. The unit of acceleration is to be compared with gravity to give for instance on the Earth surface $g \simeq 9.8$ m/s² or $g = 35.28$ Km/min².

Changing the units of measurement changes the resulting numerical values of t_c , v_c , h , m and g . In particular, if the unit of length is decreased by a factor L , the unit of time is decreased by a factor T , and the unit of mass is decreased by a factor M , the numbers h , t_c and m will increase by a factor L , T , and M , respectively (e.g., changing the unit of time from the minute to the second, which corresponds to decreasing it by a factor of 60, would cause the numerical value t_c to increase from $t_c = 3$ minutes to $t_c = 180$ seconds). It is easy to verify that the change of units proposed would increase the numerical value v_c by a factor LT^{-1} and the numerical value g by a factor LT^{-2} . These abstract positive numbers are the dimensions of the different physical quantities. Thus, we shall say that the dimensions of h , t_c , and m are, respectively, L , T and M , and this is represented in the form $[h] = L$, $[t_c] = T$, and $[m] = M$. Similarly, v_c has dimensions $[v_c] = LT^{-1}$ and g has dimensions $[g] = LT^{-2}$. The dimensions are always power-law monomials of the form $L^\alpha T^\beta M^\gamma$, where the constants α , β and γ are in general rational numbers.

To advance in the solution, let us consider now the quantities $(h/g)^{1/2}$ and $(gh)^{1/2}$. It is easy to see that decreasing the unit length and the unit time by factors L and T causes $(h/g)^{1/2}$ and $(gh)^{1/2}$ to increase by a factor T and LT^{-1} , respectively, so that we may write $[(h/g)^{1/2}] = T$ and $[(gh)^{1/2}] = LT^{-1}$. Correspondingly, the ratios

$$\Pi_1 = \frac{t_c}{(h/g)^{1/2}} \quad (5.2)$$

and

$$\Pi_2 = \frac{v_c}{(gh)^{1/2}} \quad (5.3)$$

have dimensions $[\Pi_1] = 1$ and $[\Pi_2] = 1$. They are dimensionless quantities. Their fundamental property is that their numerical value is independent of the specific system of units employed (i.e., changing the units of length and time may change t_c and $(h/g)^{1/2}$, but it leaves the value

of $t_c/(h/g)^{1/2}$ unperturbed). Using the presumed equations (5.1) we can now write

$$\Pi_1 = \frac{t_c(m, h, g)}{(h/g)^{1/2}} = \Pi_1(m, h, g) \quad (5.4)$$

and

$$\Pi_2 = \frac{v_c(m, h, g)}{(gh)^{1/2}} = \Pi_2(m, h, g) \quad (5.5)$$

for the parametric dependences of Π_1 and Π_2 . Note that Π_1 and Π_2 are dimensionless quantities, independent therefore of the system of fundamental units employed, whereas m , h , and g are dimensional quantities, whose numerical value changes when we change the units of measurement according to $[m] = M$, $[h] = L$, and $[g] = LT^{-2}$. This can be used to simplify the functional dependences given above. In particular, by modifying only the unit of mass, while leaving the units of length and time unaltered, one may change the value of m arbitrarily while maintaining the numbers h and g constant. Since Π_1 and Π_2 are independent of the system of units - and therefore remain constant when the unit of mass changes- it is clear that the presumed functional dependences (5.4) and (5.5) are incorrect, in that Π_1 and Π_2 are not a function of m , so that we reduce the number of independent variables to give $\Pi_1 = \Pi_1(h, g)$ and $\Pi_2 = \Pi_2(h, g)$. We continue the reasoning by decreasing the unit of time by a factor T , which increases the numerical value of g in a factor T^{-2} , while h (and of course Π_1 and Π_2) remains unaltered. We can then conclude that Π_1 and Π_2 are not a function of g either. Finally, by modifying the unit of length, we can change arbitrarily h and, again, that does not result in changes of Π_1 and Π_2 , so that these two quantities are constant, independent of all the presumed parameters, as expressed in

$$\frac{t_c}{(h/g)^{1/2}} = C_1 \quad (5.6)$$

and

$$\frac{v_c}{(gh)^{1/2}} = C_2. \quad (5.7)$$

Dimensional analysis alone leads to (5.6) and (5.7), but does not provide the values of C_1 and C_2 . They could be obtained, for instance, from numerical integration of Newton's second law, as shown above, giving $C_1 = C_2 = \sqrt{2}$. They could also be determined by carrying out a single experiment, by dropping a given point object of mass m from a given height h in a given planet with gravity g . Measuring the resulting impact velocity and collision time and using (5.6) and (5.7) gives then the values of C_1 and C_2 . Note that, once they are determined, the laws given in (5.6) and (5.7) serve to describe the fall of point objects of any mass, from any height on any planet. This is an example of how arguments based on the concept of dimension, possibly together with a limited number of experiments (one in this case) may determine the solution of an engineering problem. As shown below, the methodology used can be generalized, providing a rigorous procedure for the analysis of any given physical problem.

Physical quantities with independent dimensions

The number of units in the fundamental system of units depends on the character of the problem. In a geometrical problem, it is enough to introduce a unit of length, whereas for a kinematic problem we need to add one unit to measure time. In mechanics, the system includes three units (for mass, length and time, for instance), and a larger number is required for describing

thermal problems, when an additional unit must be introduced to measure temperature, or electromagnetic problems, involving a unit of electric charge. The presentation below will focus on problems of mechanical character, for which the dimensions of all physical quantities admit a power-law monomial representation of the form $[a] = L^\alpha T^\beta M^\gamma$.

We say that the k physical quantities a_1, \dots, a_k have independent dimensions when the dimensions of none of them can be expressed as a product of powers of the dimensions of the remaining $k - 1$ quantities. For instance, consider the set of three physical quantities formed by the density ($[\rho] = ML^{-3}$), velocity ($[U] = LT^{-1}$) and viscosity ($[\mu] = ML^{-1}T^{-1}$). Do they have independent dimensions? To find the answer, we begin by assuming the converse. If their dimensions were dependent we could find two numbers x and y such that $[\rho] = [U]^x[\mu]^y$. Substituting the dimensions of the different quantities yields

$$L^{-3}M^1T^0 = (L^1M^0T^{-1})^x(L^{-1}M^1T^{-1})^y, \quad (5.8)$$

which provides the three equations

$$-3 = x - y, \quad 1 = y \quad \text{and} \quad 0 = -x - y. \quad (5.9)$$

The reader can easily verify that it is not possible to find x and y satisfying simultaneously all three equations, so that we conclude that the quantities ρ , U and μ in fact have independent dimensions. As an example of three quantities that do not have independent dimensions, consider the pressure ($[p] = ML^{-1}T^{-2}$), density ($[\rho] = ML^{-3}$), and velocity ($[U] = LT^{-1}$). In this case, there exist values of $x = 1$ and $y = 2$ such that $[p] = [\rho][U]^2$.

Note that the equations (5.9) can also be expressed in the alternative form

$$(-3, 1, 0) = x(1, 0, -1) + y(-1, 1, -1), \quad (5.10)$$

revealing that ρ , U and μ have independent dimensions if the vectors $(-3, 1, 0)$, $(1, 0, -1)$ and $(-1, 1, -1)$ formed by the powers in the dimensions of each of the three quantities are linearly independent. Therefore, to check in general whether the physical quantities a_1, \dots, a_k have independent dimensions it is sufficient to see whether the rank of the matrix formed by the powers in the dimensions equals k . For the case of ρ , U and μ the corresponding matrix is

$$\begin{bmatrix} -3 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & -1 \end{bmatrix} \quad (5.11)$$

whose rank in fact equals 3, so that these three quantities have independent dimensions. However, for ρ , U and p the matrix becomes

$$\begin{bmatrix} -3 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & -2 \end{bmatrix} \quad (5.12)$$

whose rank equals 2, indicating that the three quantities ρ , U and p do not have independent dimensions.

The Π theorem (the Vaschy-Buckingham theorem)

To solve a given physical problem we need to compute an unknown physical quantity

$$a_0 = f(a_1, \dots, a_k, a_{k+1}, \dots, a_n), \quad (5.13)$$

which is a function of a number of governing parameters $a_1, \dots, a_k, a_{k+1}, \dots, a_n$. Let us assume that $k \leq n$ is the maximum number of governing parameters that have independent dimensions, with a_1, \dots, a_k denoting the subset of parameters with independent dimensions. Consequently, the dimensions of the remaining parameters a_{k+1}, \dots, a_n can be expressed in the form

$$\begin{aligned} [a_{k+1}] &= [a_1]^{\alpha_{k+1}} \dots [a_k]^{\gamma_{k+1}}, \\ [a_{k+2}] &= [a_1]^{\alpha_{k+2}} \dots [a_k]^{\gamma_{k+2}}, \\ &\vdots \\ &\vdots \\ [a_n] &= [a_1]^{\alpha_n} \dots [a_k]^{\gamma_n}. \end{aligned} \quad (5.14)$$

In general, $0 < k < n$, but sometimes $k = n$, as occurs in the analysis of the falling body presented earlier, or $k = 0$, as occurs if all parameters a_1, \dots, a_n are dimensionless. It can be shown that the dimensions of the governed parameter a_0 can also be expressed in the form

$$[a_0] = [a_1]^{\alpha_0} \dots [a_k]^{\gamma_0}. \quad (5.15)$$

We can now use (5.14) and (5.15) to define the dimensionless parameters

$$\Pi_0 = \frac{a_0}{a_1^{\alpha_0} \dots a_k^{\gamma_0}}, \Pi_1 = \frac{a_{k+1}}{a_1^{\alpha_{k+1}} \dots a_k^{\gamma_{k+1}}}, \Pi_2 = \frac{a_{k+2}}{a_1^{\alpha_{k+2}} \dots a_k^{\gamma_{k+2}}}, \dots, \Pi_{n-k} = \frac{a_n}{a_1^{\alpha_n} \dots a_k^{\gamma_n}} \quad (5.16)$$

Dividing both sides of (5.13) by $a_1^{\alpha_0} \dots a_k^{\gamma_0}$ and expressing a_0, a_{k+1}, \dots, a_n in terms of $\Pi_0, \Pi_1, \dots, \Pi_{n-k}$ provide

$$\Pi_0 = \frac{f(a_1, \dots, a_k, a_{k+1}, \dots, a_n)}{a_1^{\alpha_0} \dots a_k^{\gamma_0}} = \frac{f(a_1, \dots, a_k, \Pi_1 a_1^{\alpha_{k+1}} \dots a_k^{\gamma_{k+1}}, \dots, \Pi_{n-k} a_1^{\alpha_n} \dots a_k^{\gamma_n})}{a_1^{\alpha_0} \dots a_k^{\gamma_0}}, \quad (5.17)$$

which can alternatively be rewritten as

$$\Pi_0 = \Pi_0(a_1, \dots, a_k, \Pi_1, \dots, \Pi_{n-k}). \quad (5.18)$$

Let us illustrate the procedure with an example taken from fluid mechanics. Consider the motion of a liquid along a long pipe of circular section. The variable of interest is the pressure loss per unit length

$$P_l = -\partial p / \partial x = f(Q, D, \rho, \mu) \quad (5.19)$$

which would be a function of the volume flux to be transported Q , with $[Q] = L^3 T^{-1}$, the transverse size of the pipe (given for instance by its diameter D , with $[D] = L$) and the properties of the fluid, given by its density ($[\rho] = M L^{-3}$) and viscosity ($[\mu] = M L^{-1} T^{-1}$). In this case, therefore, $n = 4$, whereas the number of parameters that have independent dimensions is $k = 3$, as can be easily demonstrated. If we select Q, D, ρ as the subset of governing parameters with independent dimensions, then the dimensions of the remaining parameters can be expressed as

$$[\mu] = [Q]^1 [D]^{-1} [\rho]^1 \quad (5.20)$$

and

$$[P_l] = [Q]^2 [D]^{-5} [\rho]^1, \quad (5.21)$$

corresponding in this example to equations (5.14) and (5.15) in the general presentation. It is then possible to define the dimensionless quantities

$$\Pi_0 = \frac{P_l}{\rho Q^2/D^5} \quad \text{and} \quad \Pi_1 = \frac{\mu}{\rho Q/D} \quad (5.22)$$

which can be substituted into (5.19) to give

$$\Pi_0 = \frac{f(Q, D, \rho, \Pi_1 \rho Q/D)}{\rho Q^2/D^5} = F(Q, D, \rho, \Pi_1), \quad (5.23)$$

which corresponds to (5.18).

We now proceed analogously as we did in the previous analysis of the falling body, keeping in mind that the dimensionless quantities Π_0 and Π_1 are independent of the system of units utilized, while the numbers Q , D and ρ represent dimensional magnitudes whose values change when the units of length, mass and time change according to $[Q] = L^3 T^{-1}$, $[D] = L$ and $[\rho] = M L^{-3}$. Therefore, if one decreases the unit of mass by a factor M , the value of ρ would increase by that same factor, whereas the values of Π_0 , Π_1 , Q and D remain unperturbed. In that way, it is therefore possible to change arbitrarily the value of ρ , while keeping constant Π_1 , Q and D , and the change does not affect the value of Π_0 , so that Π_0 is independent of ρ and we can write $\Pi_0 = F(Q, D, \Pi_1)$ in place of (5.23). We can now continue the simplification by decreasing the unit of time by a factor T , increasing consequently the value of Q by a factor T^{-1} (recall that $[Q] = L^3 T^{-1}$), while leaving unperturbed the values of Π_0 , Π_1 and D , so that we conclude that $\Pi_0 = F(D, \Pi_1)$. Similarly, by modifying the unit of length we arrive at the final result $\Pi_0 = F(\Pi_1)$, which can be written in the form

$$\frac{P_l}{\rho Q^2/D^5} = F\left(\frac{\mu}{\rho Q/D}\right). \quad (5.24)$$

The same procedure can be used to simplify (5.18). Thus, it is always possible to find a change in the system of units such that only the value of a_1 is modified, while those of a_2, \dots, a_k remain unaltered. Since $\Pi_0, \Pi_1, \dots, \Pi_{n-k}$ are dimensionless quantities independent of the system of fundamental units, we see that modifying the governing parameter a_1 with all remaining governing parameters kept constant leads to no changes in Π_0 , so that Π_0 is not a function of a_1 . Proceeding sequentially it is possible to prove that the dimensionless quantity Π_0 is independent a_2, \dots, a_k , thereby simplifying the parametric dependence to

$$\Pi_0 = F(\Pi_1, \dots, \Pi_{n-k}). \quad (5.25)$$

The simplification of (5.13) to give (5.25) is formally stated in the so-called Π theorem: **For a given physical problem that can be expressed in the form (5.13), that is, in which we seek to find an unknown quantity a_0 as a function of the known values of its n different governing parameters a_1, \dots, a_n , it is always possible to reduce to $n - k$ the number of governing parameters, with k representing the number of governing parameters with independent dimensions. To write the simplified parametric expression, we use the k magnitudes with independent dimensions to express the remaining $n - k$ governing parameters and the unknown quantity a_0 in dimensionless form, yielding as result (5.25).**

Note that the simplification can be quite remarkable. For instance, for the flow in a pipe, application of the Π theorem allows us to reduce the number of parameters from four, i.e., (5.19),

to only one, i.e., (5.24). This simplification drastically reduces the number of experiments to be performed to characterize a given physical phenomenon. Thus, to determine the pressure loss per unit length P_l one could carry out a single series of experiments varying the flux Q and measuring the accompanying pressure loss P_l obtained in a given experimental setup with a given fluid. Representing now the results in the dimensionless form (5.24) we would have obtained the general law for pressure loss of any fluid in any pipe. To compute the pressure loss corresponding to a volume flux Q^* of a different fluid of density ρ^* and viscosity μ^* circulating along a different pipe of diameter D^* one begins by computing the value of the dimensionless governing parameter $\Pi_1^* = \mu^*/(\rho^*Q^*/D^*)$. Looking up in the plot $\Pi_0 = F(\Pi_1)$ we could then obtain the corresponding value of $\Pi_0^* = P_l^*/(\rho^*(Q^*)^2/(D^*)^5)$, which could be readily solved to determine the pressure loss per unit length $P_l^* = \Pi_0^*\rho^*(Q^*)^2/(D^*)^5$ corresponding to the specific conditions of interest.

Parametric dependence of the solution

Perhaps one of the most complicated steps for the successful application of the Π theorem is the correct identification of the governing parameters that influence the solution, that is, the writing of the starting equation (5.13). Considering too many parameters or missing an important one leads to incorrect results. While an educated guess is the only way to find the parametric dependence in some physical or engineering problems, in fluid mechanics there is always a simpler approach, consisting in writing the conservation equations with initial and boundary conditions that determine the solution. By inspection of the resulting mathematical problem one can then easily identify the governing parameters. Let us illustrate the method with an example.

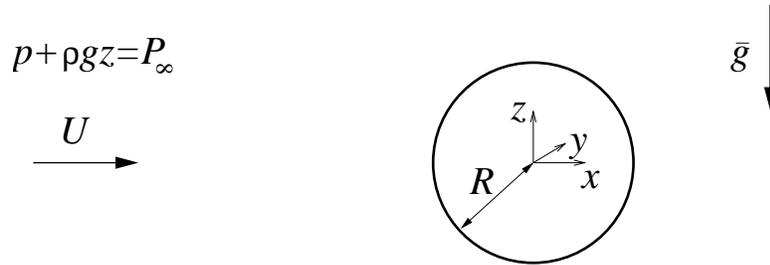


Figure 5.2: Fluid motion around a sphere.

Let us analyze the drag force F_x acting on a solid sphere moving with constant velocity U in an otherwise stagnant liquid of density ρ and viscosity μ whose pressure far from the sphere is given by the hydrostatic distribution $p + \rho g z = P_\infty$ in terms of the ambient reduced pressure P_∞ . In using the Π theorem, it might at first seem reasonable to assume the following parametric dependence

$$F_x = F_x(R, U, \rho, \mu, P_\infty, g). \quad (5.26)$$

To identify the maximum number of governing parameters with independent dimension we write

the matrix

$$\begin{array}{c}
 R \\
 U \\
 \rho \\
 \mu \\
 p_\infty \\
 g
 \end{array}
 \begin{array}{c}
 L \quad M \quad T \\
 \left[\begin{array}{ccc}
 1 & 0 & 0 \\
 1 & 0 & -1 \\
 -3 & 1 & 0 \\
 -1 & 1 & -1 \\
 -1 & 1 & -2 \\
 1 & 0 & -2
 \end{array} \right]
 \end{array}
 \quad (5.27)$$

for the powers of the dimensions $[R] = L$, $[U] = LT^{-1}$, $[\rho] = ML^{-3}$, $[\mu] = ML^{-1}T^{-1}$, $[P_\infty] = ML^{-1}T^{-2}$, and $[g] = LT^{-2}$. It is easy to see that in this case $k = 3$, by verifying for instance that the determinant of the matrix formed by the first three rows is nonzero (note that, the rank of this 6×3 matrix can be at the most $k = 3$, equal to the number of columns). We can then select (R, U, ρ) as the subset (a_1, \dots, a_k) of parameters with independent dimensions to be used for application of the Π theorem. According to what we have seen earlier, to derive the simplified expression (5.25), we have to use (a_1, \dots, a_k) to define the dimensionless quantities Π_0 and Π_1, \dots, Π_{n-k} . For instance, in the definition of a dimensionless force we use $[F_x] = [\rho][U]^2[R]^2$ to give

$$\Pi_0 = \frac{F_x}{\rho U^2 R^2}. \quad (5.28)$$

Similarly, since $[\mu] = [\rho][U][R]$, $[P_\infty] = [\rho][U]^2$, and $[g] = [U]^2[R]^{-1}$ the dimensionless parameters associated with the viscosity, ambient reduced pressure, and gravity are

$$\Pi_1 = \frac{\mu}{\rho U R}, \quad \Pi_2 = \frac{P_\infty}{\rho U^2}, \quad \text{and} \quad \Pi_3 = \frac{g}{U^2/R} \quad (5.29)$$

so that, finally, the application of the Π theorem leads to

$$\frac{F_x}{\rho U^2 R^2} = f \left(\frac{\mu}{\rho U R}, \frac{P_\infty}{\rho U^2}, \frac{g}{U^2/R} \right). \quad (5.30)$$

As can be seen, we have reduced the number of parameters from six to only three. The result is however incorrect, in that the initial guess (5.26) includes two spurious parameters P_∞ and g that do not influence the solution and should not have been included in the first place. The application of the Π theorem to the correct initial functional dependence $F_x = F_x(R, U, \rho, \mu)$ yields

$$\frac{F_x}{\rho U^2 R^2} = f \left(\frac{\mu}{\rho U R} \right), \quad (5.31)$$

instead of (5.30). It is however not that easy to anticipate that the solution does not depend on the ambient pressure and gravity, unless we proceed as suggested above by writing the conservation equations with boundary conditions that determine the drag force prior to writing (5.26). In a reference frame moving with the sphere the computation of the pressure and velocity in the fluid requires integration of

$$\begin{array}{l}
 \nabla \cdot \bar{v} = 0 \\
 \rho \bar{v} \cdot \nabla \bar{v} = -\nabla p + \mu \nabla^2 \bar{v} - \rho g \bar{e}_z
 \end{array}
 \quad \left\{ \begin{array}{l}
 |\bar{x}| \rightarrow \infty : p = P_\infty - \rho g z, \bar{v} = U \bar{e}_x \\
 |\bar{x}| = R : \bar{v} = 0
 \end{array} \right. \quad (5.32)$$

Once p and \bar{v} are determined, the total force on the sphere would be computed from

$$\bar{F} = - \int_{|\bar{x}|=R} p \bar{n} d\sigma + \int_{|\bar{x}|=R} \mu (\nabla \bar{v} + \nabla \bar{v}^T) \cdot \bar{n} d\sigma, \quad (5.33)$$

where $\bar{\tau}' = \mu(\nabla\bar{v} + \nabla\bar{v}^T)$, with the drag obtained as the x component according to

$$F_x = \bar{F} \cdot \bar{e}_x = - \int_{|\bar{x}|=R} p n_x d\sigma + \int_{|\bar{x}|=R} \mu [(\nabla\bar{v} + \nabla\bar{v}^T) \cdot \bar{n}] \cdot \bar{e}_x d\sigma. \quad (5.34)$$

A quick look at (5.32) and (5.34) indicates that, as stated in (5.26), the drag force is indeed a function of ρ , U , R and μ , but also a function of P_∞ and g . This last dependences can be however eliminated by using the reduced pressure $p + \rho g z$ to incorporate the effect of gravity according to $-\nabla p - \rho g \bar{e}_z = -\nabla(p + \rho g z)$ and noting that the effect of the pressure enters in (5.32) and (5.34) through the relative pressure differences, while the ambient level P_∞ is inconsequential. These observations suggest the use of the alternative variable $P' = p + \rho g z - P_\infty$ as a replacement for p . Rewriting (5.32) in terms of P' yields

$$\begin{cases} \nabla \cdot \bar{v} = 0 \\ \rho \bar{v} \cdot \nabla \bar{v} = -\nabla P' + \mu \nabla^2 \bar{v} \end{cases} \quad \begin{cases} |\bar{x}| \rightarrow \infty : P' = 0, \bar{v} = U \bar{e}_x \\ |\bar{x}| = R : \bar{v} = 0, \end{cases} \quad (5.35)$$

which reveals that the distributions $P'(\bar{x})$ and $\bar{v}(\bar{x})$ are independent of P_∞ and g . The force (5.33) can be expressed in terms of these two variables to give¹

$$\bar{F} = - \int_{|\bar{x}|=R} P' \bar{n} d\sigma + \int_{|\bar{x}|=R} \mu (\nabla\bar{v} + \nabla\bar{v}^T) \cdot \bar{n} d\sigma + \int_{|\bar{x}|=R} \rho g z \bar{n} d\sigma. \quad (5.36)$$

The last term corresponds to the buoyancy force associated with Archimedes law, as can be seen by using the Gauss formula to yield

$$\int_{|\bar{x}|=R} \rho g z \bar{n} d\sigma = \int_V \nabla(\rho g z) dV = \rho g \frac{4}{3} \pi R^3 \bar{e}_z, \quad (5.37)$$

and therefore gives no contribution to the drag force. The value of F_x can be therefore computed according to

$$F_x = \bar{F} \cdot \bar{e}_x = - \int_{|\bar{x}|=R} P' n_x d\sigma + \int_{|\bar{x}|=R} \mu [(\nabla\bar{v} + \nabla\bar{v}^T) \cdot \bar{n}] \cdot \bar{e}_x d\sigma \quad (5.38)$$

in terms of $P'(\bar{x})$ and $\bar{v}(\bar{x})$, independent of P_∞ and g , thereby demonstrating that $F_x = F_x(R, U, \rho, \mu)$ is the correct parametric dependence, so that the final result obtained by application of the Π theorem should be (5.31) instead of (5.30).

Note that a dependence on the pressure level appears in general in problems involving gas motion, because the absolute value of the pressure enters in the problem through the equation of state $p = \rho R_g T$. For liquid motion, a dependence on gravity appears in problems involving nonplanar free surfaces separating the liquid from the air, as occurs when waves are present. For instance, if the sphere above were moving close to the surface of the liquid (i.e., at a depth of the order of R), a boundary condition $p = p_a$ applies at the free surface $z_s = z_s(x, y)$, whose shape should be computed as part of the integration. The resulting drag force, although independent of p_a , is however a function of g (and also of the depth). The reader can verify that, in that case, using the reduced pressure in the place of p does not get rid of g in the formulation of the problem.

It is also convenient to note at this point that the selection of the subset of governing parameters with independent dimensions a_1, \dots, a_k is not always unique. For instance, for the problem of

¹Note that $\int_{|\bar{x}|=R} P_\infty \bar{n} d\sigma = 0$.

the drag force on the sphere, one could have alternatively selected the subset U , R and μ , so that application of the Π theorem would have led to

$$\frac{F_x}{\mu UR} = f\left(\frac{\rho UR}{\mu}\right). \quad (5.39)$$

In principle, both expressions (5.31) and (5.39) are equally valid. The dimensionless force, which is different in the two cases, can be seen to be a function of the dimensionless parameter

$$Re = \frac{\rho UR}{\mu} \quad (5.40)$$

which is the Reynolds number of the problem.

Physical similarity

Dimensional analysis serves in particular to guide the design of experiments. Often, it is not possible to reproduce exactly in the lab all of the conditions found in a given physical phenomenon of interest. For instance, if the size involved is too big, then the model for the experiments will need to be scaled down to a smaller size that we can handle in the lab. It is then appropriate to ask for the conditions that the experiment must satisfy to reproduce reliably the phenomenon occurring at the real large scale, and also how the experimental measurements must be adapted to yield the quantities of interest.

Let us consider again the moving sphere. In particular, assume we want to compute the drag force F_1 acting on a sphere of radius R_1 that moves with velocity U_1 in an oil of density ρ_1 and viscosity μ_1 . To obtain the result, we propose to carry out an experiment in a hydrodynamic tunnel. In designing the experiment, we need to take into account that the density and viscosity of water are different from those of the oil. Besides, the only spheres available have a radius $R_2 \neq R_1$.

To design the experiment, we shall use dimensional analysis. According to what we have seen above, the dependence of the force on the different parameters can be expressed in the form (5.31), indicating that the value of $F_x/(\rho U^2 R^2)$ is only a function of $Re = \rho UR/\mu$. In other words, if we get the Reynolds number in the experiment $Re_2 = \rho_2 U_2 R_2/\mu_2$ to match exactly the value $Re_1 = \rho_1 U_1 R_1/\mu_1$ corresponding to the real phenomenon, then the resulting values of $F_1/(\rho_1 U_1^2 R_1^2)$ and $F_2/(\rho_2 U_2^2 R_2^2)$ will also be exactly equal. With this information, we can now proceed to design the experiment.

For the laboratory experiment to be physically similar to the real phenomenon we need to impose the condition $Re_1 = Re_2$, which provides the value

$$U_2 = \frac{\mu_2 \rho_1 R_1}{\mu_1 \rho_2 R_2} U_1. \quad (5.41)$$

to be used in the experiment. The resulting measurement in the lab gives the force F_2 acting on the experimental model. The condition $Re_1 = Re_2$ guarantees that

$$\frac{F_1}{\rho_1 U_1^2 R_1^2} = \frac{F_2}{\rho_2 U_2^2 R_2^2}, \quad (5.42)$$

which can be used finally to determine the value

$$F_1 = \frac{\rho_1 U_1^2 R_1^2}{\rho_2 U_2^2 R_2^2} F_2 \quad (5.43)$$

for the force acting on the sphere moving in oil.

The procedure can be generalized. Dimensional analysis shows that the parametric dependence of the unknown value of the physical quantity

$$a_0 = f(a_1, \dots, a_k, a_{k+1}, \dots, a_n),$$

where k is the maximum number of governing parameters with independent dimensions, can be expressed in the simplified form

$$\Pi_0 = F(\Pi_1, \dots, \Pi_{n-k}).$$

When designing an experiment, it is not necessary to reproduce exactly in the laboratory the values of a_1, \dots, a_n . Instead, for the experiment to reproduce reality, or more specifically, for the experiment to be physically similar to the real phenomenon, it is only necessary that the values of the different dimensionless parameters Π_1, \dots, Π_{n-k} be equal in the experiment and in the real phenomenon. If physical similarity is satisfied, it is then possible to determine the value of a_0 in the real phenomenon from the measurement of a_0 in the lab by equating the corresponding values of Π_0 . Note that at times it is not possible to design the experiment in such a way that all of the governing parameters Π_1, \dots, Π_{n-k} take on equal values. In that case, physical similarity is only partially satisfied. For the experiment to be successful, we need to ensure the equality of those parameters anticipated to be most influential for the problem at hand, while permitting that parameters of secondary importance take on different values.

Nondimensionalization of the Navier-Stokes equations

An alternative to the Π theorem that allows us to identify the minimum number of parameters governing a given fluidmechanical problem consists in rewriting the mathematical problem in dimensionless form by introducing appropriately rescaled dependent and independent variables. The scales to be used depend on the initial and boundary conditions of the problem. For instance, for the motion over a sphere, the velocity U is the appropriate scale to define a dimensionless velocity $\bar{v}^* = \bar{v}/U$, while the radius of the sphere R introduces a natural length to scale the spatial coordinate, yielding $\bar{x}^* = \bar{x}/R$, $\nabla^* = R\nabla$ and $d\sigma^* = R^{-2}d\sigma$. In principle, it is not easy to find in this case a characteristic value for the pressure difference Δp to be used in scaling $P' = p + \rho gz - P_\infty$. This value, in fact, depends on the character of the fluid motion. To see this, it is convenient to estimate the order of magnitude of each one of the terms appearing in the momentum equation

$$\begin{aligned} \rho\bar{v} \cdot \nabla\bar{v} &= -\nabla P' + \mu\nabla^2\bar{v}. \\ \rho U^2/R &\quad \Delta p/R \quad \mu U/R^2 \end{aligned} \tag{5.44}$$

To estimate the order of magnitude of the inertial term $O(\rho\bar{v} \cdot \nabla\bar{v}) \sim \rho U^2/R$ we have taken into account that the characteristic velocity is of order $O(\bar{v}) \sim U$ and that in the flow field we observe velocity differences of order U when traversing distances of order R , so that $O(\nabla\bar{v}) \sim U/R$. The estimate of the pressure gradient $O(\nabla P') \sim \Delta p/R$ is written using the unknown characteristic value Δp of the pressure differences in the flow field. Similarly, the order of magnitude of the resultant of the viscous stresses per unit volume is $O(\mu\nabla^2\bar{v}) \sim \mu U/R^2$.

The above estimates enable the relative importance of the different terms to be easily anticipated. We know from experience that the pressure gradient is almost always essential for the fluid

motion², so that the remaining question is to clarify the relative importance of the convective acceleration $\rho\bar{v} \cdot \nabla\bar{v}$ and the resultant of the viscous forces $\mu\nabla^2\bar{v}$ by comparing their orders of magnitude according to

$$\frac{O(\rho\bar{v} \cdot \nabla\bar{v})}{O(\mu\nabla^2\bar{v})} = \frac{\rho UR}{\mu}. \quad (5.45)$$

As can be seen, the dimensionless number that measures the relative importance of inertial (convective) effects and viscosity is the so-called Reynolds number, Re , defined above in (5.40). Given the sphere radius, its velocity and the fluid properties, one may easily determine the value of Re . For the motion of fluids at $Re \sim O(1)$, acceleration and viscous stresses have comparable effects so that in principle one needs to retain all three terms in (5.44) for describing the solution. On the other hand, if $Re \gg 1$, the effect of viscosity can be anticipated to be negligible, so that (5.44) simplifies to the balance $\rho\bar{v} \cdot \nabla\bar{v} = -\nabla P'$ between acceleration and pressure forces. Conversely, if $Re \ll 1$ the flow has negligible acceleration, and the motion established responds to a local balance between pressure and viscous forces according to $0 = -\nabla P' + \mu\nabla^2\bar{v}$. These results can be used to evaluate the characteristic value of Δp depending on the value of Re . Thus, for fluid motion at large Reynolds numbers, using the condition $O(\rho\bar{v} \cdot \nabla\bar{v}) \sim O(\nabla P')$ gives $\Delta p \sim \rho U^2$, whereas for $Re \ll 1$ the value $\Delta p \sim \mu U/R$ follows from $O(\mu\nabla^2\bar{v}) \sim O(\nabla P')$. Note that, for $Re \sim 1$ both estimates are equivalent. Clearly, the selection for Δp depends on the character of the problem. For $Re \gtrsim 1$ the appropriate choice is $p^* = (p + \rho gz - P_\infty)/(\rho U^2)$, reducing (5.36) to

$$\nabla^* \cdot \bar{v}^* = 0, \quad \bar{v}^* \cdot \nabla^* \bar{v}^* = -\nabla^* p^* + \frac{1}{\mathbf{Re}} \nabla^{*2} \bar{v}^* \begin{cases} |\bar{x}^*| \rightarrow \infty : p^* = 0, \bar{v}^* = \bar{e}_x \\ |\bar{x}^*| = 1 : \bar{v}^* = 0 \end{cases} \quad (5.46)$$

and

$$\frac{F_x}{\rho U^2 R^2} = - \int_{|\bar{x}^*|=1} p^* \bar{n} \cdot \bar{e}_x d\sigma^* + \frac{1}{\mathbf{Re}} \int_{|\bar{x}^*|=1} [(\nabla^* \bar{v}^* + \nabla^{*T} \bar{v}^*) \cdot \bar{n}] \cdot \bar{e}_x d\sigma^*. \quad (5.47)$$

As anticipated before, by writing the problem in dimensionless form we are able to reduce to the minimum the set of governing parameters that influence the solution. In this case, observation of (5.47) indicates that the value of $F_x/(\rho U^2 R^2)$ is only a function of the dimensionless parameter $Re = \rho UR/\mu$, exactly the same result (5.31) that we previously obtained by application of the Π theorem.

On the other hand, when $Re \lesssim 1$ it is more convenient to use $\Delta p \sim \mu U/R$ to scale the pressure variations $p^* = (p + \rho gz - P_\infty)/(\mu U/R)$, which reduces the problem (5.36) to the dimensionless alternative form

$$\nabla^* \cdot \bar{v}^* = 0, \quad \mathbf{Re} \bar{v}^* \cdot \nabla^* \bar{v}^* = -\nabla^* p^* + \nabla^{*2} \bar{v}^* \begin{cases} |\bar{x}^*| \rightarrow \infty : p^* = 0, \bar{v}^* = \bar{e}_x \\ |\bar{x}^*| = 1 : \bar{v}^* = 0 \end{cases} \quad (5.48)$$

and

$$\frac{F_x}{\mu UR} = - \int_{|\bar{x}^*|=1} p^* \bar{n} \cdot \bar{e}_x d\sigma^* + \int_{|\bar{x}^*|=1} [(\nabla^* \bar{v}^* + \nabla^{*T} \bar{v}^*) \cdot \bar{n}] \cdot \bar{e}_x d\sigma^*. \quad (5.49)$$

In this case, the dimensionless problem indicates that $F_x/(\mu UR)$ is only a function of Re , the same result we obtained previously in (5.39).

²Note that, if the pressure gradient were not important, then the mathematical problem (5.35) would involve three unknowns (the three components of the velocity) but four equations (continuity plus the three components of momentum) and would be therefore overdetermined.

Selection of parameters with independent dimensions

As discussed above, there is often a certain degree of arbitrariness when selecting the subset of parameters a_1, \dots, a_k , in that different choices are available. The selection must be guided by the general principle that the parameters selected must have a significant influence for the flow conditions considered. For instance, for the motion over a sphere, one can anticipate that the force will increase significantly with increasing values of R and U , so that it is reasonable to use these two parameters as part of the subset a_1, \dots, a_k . On the other hand, in cases where the viscosity is anticipated to play an important role, the third parameter should be μ , which leads to (5.39). On the contrary, if the viscosity is not foreseen as having a significant influence, then it is more appropriate to use ρ as the third parameter with independent dimensions, yielding (5.31) after simplification.

To anticipate the relative importance of a given parameter, we often have to use our intuition. In fluid mechanics, the precise knowledge of the mathematical framework describing fluid motion simplifies things. For instance, to see whether the viscosity is important in a given problem one could evaluate the Reynolds number, given in general by $Re = \rho U_c L_c / \mu$, where U_c and L_c are the characteristic values of the fluid velocity and flowfield size. If the scales are such that $Re \lesssim 1$, then we can anticipate that the viscosity will be important, so that μ is to be selected as part of the subset a_1, \dots, a_k , while discarding the density. Conversely, when $Re \gtrsim 1$ we expect the viscosity to have a secondary role, and the appropriate subset of parameters with independent dimensions should be (U_c, L_c, ρ) (with $U_c = U$ and $L_c = R$ for the flow over a sphere).

The selection is more justified for extreme values of Re . For instance, for $Re \gg 1$ one can conclude from (5.46) and (5.47) that the effect of viscosity will be entirely negligible, so that $F_x / (\rho U R^2)$ becomes independent of Re , which implies that (5.31) simplifies to give

$$\lim_{Re \gg 1} \frac{F_x}{\rho U^2 R^2} = C_1, \quad (5.50)$$

where C_1 is a constant. Similarly, in view of (5.48) and (5.49), it is clear that for $Re \ll 1$ the value of $F_x / (\mu U R)$ becomes independent of Re , yielding

$$\lim_{Re \ll 1} \frac{F_x}{\mu U R} = C_2, \quad (5.51)$$

as a simplification of (5.39) for this case of negligibly small Reynolds number. As can be seen, the parameter Re ceases to exert an influence on the solution when it takes on extreme values, provided that in the analysis we have made an appropriate choice for a_1, \dots, a_k , i.e., to analyze the limit $Re \gg 1$ we need to anticipate that the effect of viscosity is negligible in selecting $a_1, \dots, a_k = \rho, U$ and R , whereas to analyze the limit $Re \ll 1$ the dominant effect of the viscosity should be accounted for when selecting $a_1, \dots, a_k = \mu, U$ and R as an appropriate subset of governing parameters with independent dimensions.

The solution would not be independent of Re if the selection of a_1, \dots, a_k is not appropriate. For instance, for small values of Re the limiting value of $f(Re)$ in (5.31) does not become independent of Re . Instead, one obtains

$$\lim_{Re \ll 1} \frac{F_x}{\rho U^2 R^2} = C_2 Re^{-1}, \quad (5.52)$$

as can be concluded in view of (5.51). Correspondingly, the limiting value of (5.39) for $Re \gg 1$ leads to

$$\lim_{Re \gg 1} \frac{F_x}{\mu U R} = C_1 Re, \quad (5.53)$$

as can be deduced from (5.50).

In view of these results, it is possible to make some general comments concerning the simplification of (5.25)

$$\Pi_0 = F(\Pi_1, \dots, \Pi_{n-k})$$

arising when one of the parameters Π_1, \dots, Π_{n-k} takes on extreme values (very small or very large). If for instance $\Pi_1 \gg 1$, it often occurs that Π_0 no longer depends on this parameter, so that

$$\lim_{\Pi_1 \gg 1} \Pi_0(\Pi_1, \dots, \Pi_{n-k}) = f(\Pi_2, \dots, \Pi_{n-k}). \quad (5.54)$$

In this case, we reduce the number of parameters in one. Other times, the limiting value of Π_0 continues depending on the large (or small) parameter Π_1 , but the dependence takes the simplified form

$$\lim_{\Pi_1 \gg 1} \Pi_0(\Pi_1, \dots, \Pi_{n-k}) = \Pi_1^\lambda f(\Pi_2, \dots, \Pi_{n-k}), \quad (5.55)$$

where λ is a given number, to be determined. As we have seen above, an appropriate selection of a_1, \dots, a_k often simplifies the treatment of limiting cases, leading to expressions of the form (5.54).

Taylor's analysis of a nuclear explosion

In an atomic explosion a large amount of energy E is released almost instantaneously at a localized position. In the initial stages of explosion development, a very strong spherical shock wave is formed whose radius r_o increases with time t . In the beginning, the pressure p_o behind the shock is several thousands times larger than the ambient value, whose influence is therefore negligible, reducing in this case the general parametric dependence of the solution (5.13) to

$$r_o = f_1(E, \rho_a, t) \quad \text{and} \quad p_o = f_2(E, \rho_a, t). \quad (5.56)$$

The three governing parameters E , ρ_a , and t have independent dimensions, as can be verified for instance by computing the rank of the matrix

$$\begin{array}{c} E \\ \rho_a \\ t \end{array} \begin{array}{ccc} L & M & T \\ \left[\begin{array}{ccc} 2 & 1 & -2 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array} \quad (5.57)$$

formed by the powers appearing in the dimension functions $[E] = ML^2T^{-2}$, $[\rho_a] = ML^{-3}$ and $[t] = T$. According to the Π theorem, we can in this case reduce the number of governing parameters from $n = 3$ to $n - k = 3 - 3 = 0$. Since $[r_o] = L = [E]^{1/5}[\rho_a]^{-1/5}[t]^{2/5}$ and $[p_o] = ML^{-1}T^{-2} = [E]^{2/5}[\rho_a]^{3/5}[t]^{-6/5}$, the solution simplifies to

$$\frac{r_o}{E^{1/5}\rho_a^{-1/5}t^{2/5}} = C_1 \quad \text{and} \quad \frac{p_o}{E^{2/5}\rho_a^{3/5}t^{-6/5}} = C_2, \quad (5.58)$$

which can also be written as

$$r_o = C_1 E^{1/5} \rho_a^{-1/5} t^{2/5} \quad \text{and} \quad p_o = C_2 E^{2/5} \rho_a^{3/5} t^{-6/5}. \quad (5.59)$$

The constants C_1 and C_2 must be determined by a separate analysis of compressible flow, involving integration of the corresponding conservation equations, or by experimental measurement. The shock wave progressively weakens as time goes by, so that its propagation velocity $dr_o/dt = (2/5)C_1 E^{1/5} \rho_a^{-1/5} t^{-3/5}$ and the overpressure p_o given in (5.59) continuously decrease. The solution is no longer valid as the overpressure decays to values of the order of the atmospheric pressure p_a , which occurs for $t \sim E^{1/3} \rho_a^{1/2} / p_a^{5/6}$. The reader can use these ideas to analyze the shock wave induced by a lightning bolt, which can be assumed to be equivalent to the instantaneous deposition along a line of a large amount of energy, with E representing the amount of energy released per unit length ($[E] = MLT^{-2}$).