Chapter 2

Flow kinematics

Vector and tensor formulae

This introductory section presents a brief account of different definitions of vector and tensor analysis that will be used in the following chapters. In the following, we shall consider an orthogonal coordinate system with unit vectors ($\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$) at each point. In that coordinate system, any vector $\bar{a}$ admits a representation as a function of three scalars a_i such that $\bar{a} = \sum_i a_i \bar{e}_i$. Some common operations between two vectors $\bar{a}$ and $\bar{b}$ include the **scalar product** $\bar{a} \cdot \bar{b} = \sum_i a_i b_i$ and the **cross product**

$$\bar{a} \wedge \bar{b} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\bar{e}_1 + (a_3 b_1 - a_1 b_3)\bar{e}_2 + (a_1 b_2 - a_2 b_1)\bar{e}_3. \tag{2.1}$$

A second-order tensor $\bar{\bar{A}}$ is to be interpreted in the framework of the present course as a linear operator that, when acting over a vector $\bar{a}$, gives as result another vector $\bar{b} = \bar{\bar{A}} \cdot \bar{a}$ with components $b_i = \sum_j A_{ij} a_j$. The transverse tensor $\bar{\bar{A}}^{\mathrm{T}}$, with components $(\bar{\bar{A}}^{\mathrm{T}})_{ij} = (\bar{\bar{A}})_{ji}$, satisfies $\bar{a} \cdot \bar{\bar{A}} = \bar{\bar{A}}^{\mathrm{T}} \cdot \bar{a}$. The tensor $\bar{\bar{\mathrm{I}}}$ with components $\mathrm{I}_{ii} = 1$ y $\mathrm{I}_{ij} = 0$ if $i \neq j$ is the so-called identity tensor, which verifies $\bar{\bar{\mathrm{I}}} \cdot \bar{\mathrm{a}} = \bar{\mathrm{a}}$ for any given vector $\bar{a}$. Of interest in the following development are the **dyadic product of two vectors**, $\bar{a}\bar{b}$, which produces a tensor of components $(\bar{a}\bar{b})_{ij} = a_i b_j$, and the **contraction of two tensors**, $\bar{\bar{A}} : \bar{\bar{B}}$ which gives as a result a scalar $\bar{\bar{A}} : \bar{\bar{B}} = \sum_i \sum_j A_{ij} B_{ij}$. Note that this last result is quite different from the product of the two tensors, $\bar{\bar{A}} \cdot \bar{\bar{B}}$, which is another tensor of components $(\bar{\bar{A}} \cdot \bar{\bar{B}})_{ij} = \sum_k A_{ik} B_{kj}$.Also of interest is the **cross product of a vector** $\bar{a} = (a_1, a_2, a_3)$ **and a tensor** $\bar{\bar{A}}$, which gives a tensor according to

$$\bar{a} \wedge \bar{\bar{A}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \bar{\bar{A}}. \tag{2.2}$$

Besides cartesian (rectangular) coordinates ($x_1 = x$, $x_2 = y$, $x_3 = z$), two other types of orthogonal curvilinear coordinates will be found to be useful: cylindrical coordinates ($x_1 = r$, $x_2 = \theta$, $x_3 = z$) and spherical coordinates ($x_1 = r$, $x_2 = \theta$, $x_3 = \phi$), which are schematically represented in Fig. 2.1, along with their associated unit vectors ($\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$).

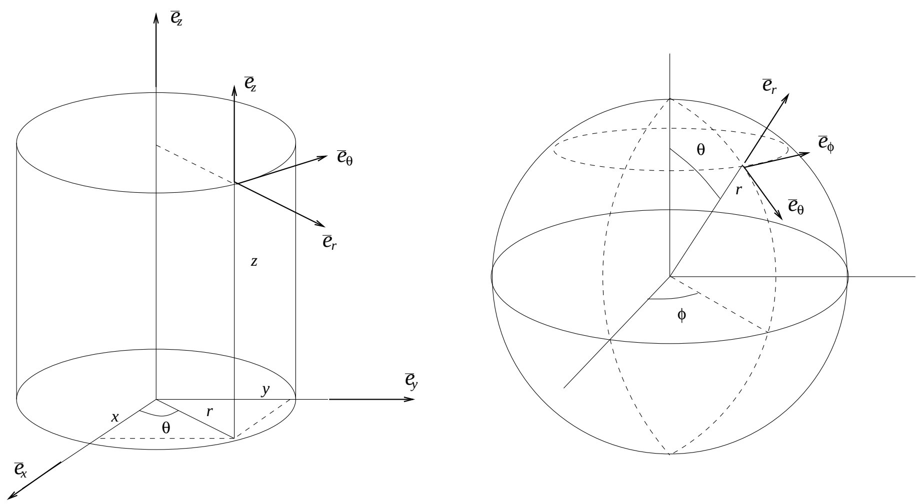

Figure 2.1: Schematic representation of relevant coordinate systems.

The differential line elements associated for each of the three coordinate systems considered are, respectively, $\mathrm{d}\bar{x} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$, $\mathrm{d}\bar{x} = (\mathrm{d}r, r\mathrm{d}\theta, \mathrm{d}z)$ and $\mathrm{d}\bar{x} = (\mathrm{d}r, r\mathrm{d}\theta, r\sin\theta\mathrm{d}\phi)$. The coefficients that premultiply each of the differentials in the above expressions are the so-called scale factors (h_1, h_2, h_3), which reduce to $(h_1 = 1, h_2 = 1, h_3 = 1)$ for cartesian coordinates, to $(h_1 = 1, h_2 = r, h_3 = 1)$ for cylindrical coordinates, and to $(h_1 = 1, h_2 = r, h_3 = r\sin\theta)$ for spherical coordinates. The introduction of these scale factors enables the differential operators to be written in a compact form independent of the coordinate system. Thus, the **gradient of a scalar function** Φ is in general given by

$$\nabla\Phi = \left(\frac{1}{h_1}\frac{\partial\Phi}{\partial x_1}, \frac{1}{h_2}\frac{\partial\Phi}{\partial x_2}, \frac{1}{h_3}\frac{\partial\Phi}{\partial x_3}\right), \tag{2.3}$$

while its **laplacian** can be expressed in the form

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial x_3}\right)\right]. \tag{2.4}$$

Similarly, the **divergence of a vector function** $\bar{a} = a_1\bar{e}_1 + a_2\bar{e}_2 + a_3\bar{e}_3$ is

$$\nabla\cdot\bar{a} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3a_1) + \frac{\partial}{\partial x_2}(h_1h_3a_2) + \frac{\partial}{\partial x_3}(h_1h_2a_3)\right], \tag{2.5}$$

while its **curl** is given by

$$\nabla\wedge\bar{a} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\bar{e}_1 & h_2\bar{e}_2 & h_3\bar{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix} = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3a_3) - \frac{\partial}{\partial x_3}(h_2a_2)\right]\bar{e}_1 + \frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1a_1) - \frac{\partial}{\partial x_1}(h_3a_3)\right]\bar{e}_2 + \frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2a_2) - \frac{\partial}{\partial x_2}(h_1a_1)\right]\bar{e}_3. \tag{2.6}$$

Also, the **gradient of a vector function** produces a tensor according to

$$(\nabla\bar{a})_{ii} = \frac{1}{h_i}\frac{\partial a_i}{\partial x_i} + \sum_{k\neq i}\frac{a_k}{h_ih_k}\frac{\partial h_i}{\partial x_k} \quad \text{and} \quad (\nabla\bar{a})_{ij} = \frac{1}{h_i}\frac{\partial a_j}{\partial x_i} - \frac{a_i}{h_ih_j}\frac{\partial h_i}{\partial x_j} \quad \text{if} \quad i\neq j, \tag{2.7}$$

while the **divergence of a tensor** $\nabla \cdot \bar{\bar{A}}$ gives a vector function of components

$$(\nabla \cdot \bar{\bar{A}})_i = \frac{h_i}{h} \sum_j \frac{\partial}{\partial x_j} \left(\frac{h A_{ij}}{h_i h_j} \right) + \sum_j \frac{A_{ij} + A_{ji}}{h_i h_j} \frac{\partial h_i}{\partial x_j} - \sum_j \frac{A_{jj}}{h_i h_j} \frac{\partial h_j}{\partial x_i}, \tag{2.8}$$

where $h = h_1 h_2 h_3$.
The so-called Gauss formula

$$\int_\Sigma (\bar{n} \circ \phi) \mathrm{d}\sigma = \int_V (\nabla \circ \phi) \mathrm{d}V, \tag{2.9}$$

where ϕ represents a scalar, vector or tensor function, V is a domain enclosed by the closed surface Σ, and $\circ$ denotes any of the products defined above, will be found to be useful in manipulating the conservation equations.

Some preliminary concepts

Two different approaches, called respectively **Eulerian and Lagrangian descriptions**, can be adopted when describing the velocity field. The Lagrangian approach takes a discrete point of view by treating the fluid as an ensemble of fluid particles. The flow field is described in terms of the fluid-particle trajectories

$$\bar{x} = \bar{x}_{\mathrm{T}}(t; \bar{x}_o, t_o), \tag{2.10}$$

with each fluid particle identified by its initial position $\bar{x}_o$ at the initial time time t_o. The associated velocity and acceleration is computed by straightforward derivation to give $\bar{v} = \mathrm{d}\bar{x}_{\mathrm{T}}/\mathrm{d}t$ and $\bar{a} = \mathrm{d}^2\bar{x}_{\mathrm{T}}/\mathrm{d}t^2$. This Lagrangian approximation, which might be of use in specific applications (e.g., for the analysis of the disperse phase in two-phase problems), leads in general to complex conservation laws. For that reason, in the following derivation we shall adopt instead the Eulerian description, which assumes the velocity to be a continuous vector field $\bar{v}(\bar{x}, t)$ as a function of the position $\bar{x}$ and time t.
A flow field is said to be uniform when the spatial differences of the different flow properties are strictly zero. Similarly, a fluid flow is said to be steady when it does not change in time. Thus, a **uniform velocity** field satisfies $\bar{v} = \bar{v}(t)$, whereas whereas a **steady velocity** field is such that $\bar{v} = \bar{v}(\bar{x})$. At times, the steadiness of a given flow depends on the observer. For instance, for a flow over an aircraft moving with constant velocity, the flow is clearly unsteady for an observer standing on the ground, whereas for the pilot the flow is effectively steady. A moving frame of reference in that case would clearly facilitate the analysis of the flow.
A stagnation point is defined as a point where the velocity is zero (all three components). For a given velocity field $\bar{v} = \bar{v}(\bar{x}, t)$, the stagnation points are determined by the equation

$$\bar{v}(\bar{x}, t) = 0. \tag{2.11}$$

Clearly, the location of the stagnation points depends on the reference frame selected.

Trajectories, path lines, and streak lines

In its motion, a fluid particle initially located at $\bar{x} = \bar{x}_o$ changes its position in time as expressed in (2.10). Given the velocity field $\bar{v}(\bar{x}, t)$, the location of the fluid particle at each instant of time is determined by integration of

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \bar{v}(\bar{x}, t) \tag{2.12}$$

with initial condition $\bar{x} = \bar{x}_o$ at $t = t_o$ to give $\bar{x} = \bar{x}_{\mathrm{T}}(t; \bar{x}_o, t_o)$. The above vector equation can be alternatively expressed in its three components

$$\mathrm{d}t = \frac{h_1 \mathrm{d}x_1}{v_1(x_1, x_2, x_3, t)} = \frac{h_2 \mathrm{d}x_2}{v_2(x_1, x_2, x_3, t)} = \frac{h_3 \mathrm{d}x_3}{v_3(x_1, x_2, x_3, t)} \tag{2.13}$$

to be integrated subject to $x_1 = x_{1_o}$, $x_2 = x_{2_o}$ and $x_3 = x_{3_o}$ at $t = t_o$. The solution can be expressed in the general form (2.10), which reduces to

$$\begin{aligned} x &= x_{\mathrm{T}}(t; x_o, y_o, z_o, t_o) \\ y &= y_{\mathrm{T}}(t; x_o, y_o, z_o, t_o) \\ z &= z_{\mathrm{T}}(t; x_o, y_o, z_o, t_o) \end{aligned} \tag{2.14}$$

for cartesian coordinates.
The trajectory contains information about the **path** followed by the particle and also about the rate with which it travels along it. The equations that describe the path lines can be obtained by eliminating the time in (2.10). For instance, for cartesian coordinates, once the time has been eliminated in (2.14), the following two equations emerge

$$f(x_o, y_o, z_o, t_o, x, y, z) = g(x_o, y_o, z_o, t_o, x, y, z) = 0 \tag{2.15}$$

defining two surfaces whose intersection is the path line followed by the fluid particle initially located at (x_o, y_o, z_o).
Besides path lines, in connection with the particle trajectories it is of interest to introduce the concept of **streak lines**. At a given time $t = t^*$, consider the fluid particles that have passed through the point $\bar{x} = \bar{x}_o$ at an earlier time t_o. These particles lie on a curve, that we call **streak line**. They are useful in experiments for observational purposes; they can be visualized by injecting dye slowly at some fixed point. Its mathematical description readily follows from evaluating the equations for the particle trajectories (2.14) at a fixed time $t = t^*$ for varying t_o. Eliminating t_o from the three equations provides the alternative description $\mathcal{F}(x_o, y_o, z_o, t^*, x, y, z) = \mathcal{G}(x_o, y_o, z_o, t^*, x, y, z) = 0$ in terms of two intersecting surfaces.

Fluid lines, fluid surfaces and fluid volumes

Fluid particles initially located along the curve

$$\bar{x} = \bar{x}_l(\lambda), \tag{2.16}$$

where λ denotes a parameter describing the curve [1], will continue to form a curve at any subsequent time. The equation describing the evolution of a **fluid line** is obtained by writing the equation for the trajectories, given in (2.10), for the fluid particles located at the initial time $t = t_o$ along the curve defined in (2.16) to yield

$$\bar{x} = \bar{x}_{\mathrm{T}}(\bar{x}_l(\lambda), t). \tag{2.17}$$

[1] Recall that a curve can be in general expressed in the parametric form $\bar{x} = \bar{x}_l(\lambda)$ with $\lambda_1 < \lambda < \lambda_2$. For instance, the vertical axis is given by the expression $x = 0, y = 0, z = \lambda$ $(-\infty < \lambda < \infty)$, while the parametric expression for a circle of radius R contained on the horizontal plane and centered at the origin is $x = R\cos\lambda, y = R\sin\lambda, z = 0$ $(0 < \lambda < 2\pi)$.

Similarly, fluid particles initially forming a surface

$$\bar{x} = \bar{x}_s(\alpha, \beta), \tag{2.18}$$

where α and β are parameters[2] will continue to form a surface (**fluid surface**). The corresponding fluid-surface equation

$$\bar{x} = \bar{x}_{\mathrm{T}}(\bar{x}_s(\alpha, \beta), t), \tag{2.19}$$

can be simplified by eliminating the parameters α and β, yielding in general an implicit equation of the form

$$f(\bar{x}, t) = 0. \tag{2.20}$$

If the surface is initially closed, it can be expected to remain closed in its evolution. The finite volume of fluid bounded by a closed fluid surface is called a **fluid volume**, a useful concept to be used later when deriving the conservation laws governing the fluid motion. In particular, in view of the definition of the fluid velocity as the velocity of the center of mass of the fluid particle (see (1.4)), it is clear that mass cannot cross a fluid surface, which implies that **the mass of a fluid volume remains constant**, thereby providing the first conservation law of fluid mechanics.

Streamlines, stream surfaces and stream tubes

The **streamlines** of a flow field at a given instant of time $t = t^*$ are lines that are tangent to the local velocity vector at each point. This tangency condition can be expressed in the form

$$\frac{h_1 \mathrm{d}x_1}{v_1(x_1, x_2, x_3, t^*)} = \frac{h_2 \mathrm{d}x_2}{v_2(x_1, x_2, x_3, t^*)} = \frac{h_3 \mathrm{d}x_3}{v_3(x_1, x_2, x_3, t^*)} \tag{2.21}$$

The two integration constants for the above pair of differential equations serve to identify a specific streamline. For instance, one could identify a streamline by selecting the location (x_o, y_o) at which it crosses the horizontal plane $z = 0$. If two streamlines intersect, the crossing point is necessarily a stagnation point (why?). There are two additional concepts related to the streamlines that are worth mentioning. The surface formed by all streamlines intersecting a given line is called a **stream surface**. If the line selected is closed, the resulting stream surface would form a **stream tube**.

Although Eqs. (2.13) and (2.21) look similar, they represent two very different physical and mathematical concepts. The time t is an integration variable in the initial value problem defined by (2.13), whereas it merely plays the role of a parameter in the integration of (2.21). To find a trajectory, we follow the temporal evolution of the fluid particle. To find a streamline, we *freeze* the time by looking at the velocity field existing at a specific instant.

Path lines and streamlines differ in a general unsteady flow. However, for a steady velocity field $\bar{v} = \bar{v}(\bar{x})$ or, more generally, for a velocity field of the form $\bar{v} = f(t)\bar{V}(\bar{x})$ with $f(t)$ representing an arbitrary function of time, streamlines and path lines can be shown to coincide (show it!).

[2] In general, a surface can be defined in terms of a pair of parameters. For instance, a sphere of radius R centered at the origin admits the parametric representation $x = R\sin\alpha\cos\beta, y = R\sin\alpha\sin\beta, z = R\cos\alpha$ $(0 < \alpha < \pi, 0 < \beta < 2\pi)$.

Material derivative

To determine the variation of any intensive scalar property ϕ following the fluid flow, such as the density, pressure or temperature, one must account for the fluid motion by considering the infinitesimal displacement of a fluid particle, as illustrated in Fig. 2.2. At time t the fluid

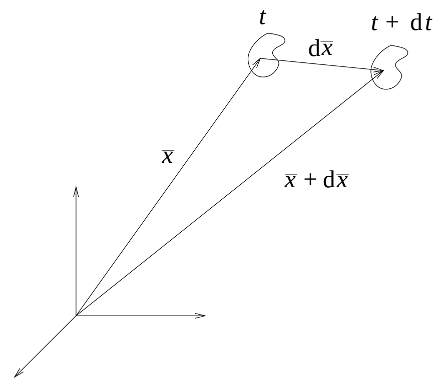

Figure 2.2: Infinitesimal displacement of a fluid particle.

particle is located at $\bar{x}$, so that the initial value of the intensive property is given by $\phi(\bar{x}, t)$. A time increment $\mathrm{d}t$ later, the particle has moved to occupy a new position given by $\bar{x} + \mathrm{d}\bar{x}$ ($\mathrm{d}\bar{x} = \bar{v}\mathrm{d}t$), and the new value of ϕ would be given by $\phi(\bar{x} + \mathrm{d}\bar{x}, t + \mathrm{d}t)$. Neglecting small terms in a Taylor expansion of ϕ about $(\bar{x}, t)$ enables the variation of ϕ for the fluid particle to be written in the approximate form

$$\mathrm{d}\phi = \phi(\bar{x} + \mathrm{d}\bar{x}, t + \mathrm{d}t) - \phi(\bar{x}, t) = \mathrm{d}t\frac{\partial \phi}{\partial t} + \mathrm{d}\bar{x} \cdot \nabla\phi. \tag{2.22}$$

Dividing the above expression by $\mathrm{d}t$ and identifying $\bar{v} = \mathrm{d}\bar{x}/\mathrm{d}t$ leads to

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial \phi}{\partial t} + \bar{v} \cdot \nabla\phi, \tag{2.23}$$

where the material derivative operator

$$\frac{\mathrm{D}()}{\mathrm{D}t} = \frac{\partial ()}{\partial t} + \bar{v} \cdot \nabla() \tag{2.24}$$

expresses the variation in time of any intensive scalar property following the fluid particle. The first term on the right-hand side represents the local temporal variation, while the second term is the so-called convective derivative, related to the motion of the fluid particle. Changes in the properties of a fluid particle may arise because the flow is unsteady and/or because the fluid particle moves through a nonuniform field [3].

[3] Note that the concept of material derivative serves to derive the differential equation to be satisfied by the function $f(\bar{x}, t)$ corresponding to the fluid surface $f(\bar{x}, t) = 0$ (see Eq. (2.20)). The fluid surface therefore divides the flow field in two regions where the sign of $f(\bar{x}, t)$ is different. Since the fluid particles defining the fluid surface maintain a constant value $f(\bar{x}, t) = 0$ in their evolution, the function $f(\bar{x}, t)$ must satisfy the equation

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \bar{v} \cdot \nabla f = 0.$$

Acceleration

The concept of material derivative can be extended to determine the time evolution of a vector property. In particular, we shall define the acceleration vector as

$$\bar{a} = \frac{\mathrm{D}\bar{v}}{\mathrm{D}t} = \frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot (\nabla \bar{v}), \tag{2.25}$$

where the gradient of velocity $\nabla \bar{v}$ is a tensor. A convenient way to express (2.25) is

$$\bar{a} = \frac{\partial \bar{v}}{\partial t} + \nabla(|\bar{v}|^2/2) - \bar{v} \wedge (\nabla \wedge \bar{v}), \tag{2.26}$$

which is valid regardless of the coordinate system selected.
In cartesian coordinates, the components of the gradient of velocity are given by $(\nabla \bar{v})_{ij} = \partial v_j/\partial x_i$, so that each component of the acceleration a_i reduces to the material derivative of the corresponding velocity component according to

$$a_i = \frac{\mathrm{D}v_i}{\mathrm{D}t} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}. \tag{2.27}$$

Note that this last result does not hold when cylindrical or spherical coordinates are used.
For the description of certain flow fields, it is at times convenient to employ a non-inertial reference frame, that is, a reference frame that is accelerating and/or rotating with respect to the *laboratory* frame of reference. In that case, when writing the fluid acceleration one must account for the additional inertial contribution

$$\bar{a}_s = \bar{a}_o + \frac{\mathrm{d}\bar{\Omega}}{\mathrm{d}t} \wedge \bar{x} + \bar{\Omega} \wedge (\bar{\Omega} \wedge \bar{x}) + 2\bar{\Omega} \wedge \bar{v}, \tag{2.28}$$

where $\bar{a}_o$ and $\bar{\Omega}$ are the acceleration and angular velocity of the non-inertial reference frame.

Circulation and vorticity

The **circulation** Γ along a curve L is defined as the integral

$$\Gamma = \int_L \bar{v} \cdot \mathrm{d}\bar{l}, \tag{2.29}$$

where the vector $\mathrm{d}\bar{l}$ represents the differential line element. Physically, the circulation gives as a measure of the fluid motion along the direction of the curve. If $\bar{x} = \bar{x}_l(\lambda)$ $(\lambda_1 < \lambda < \lambda_2)$ is the parametric representation of the curve, then

$$\mathrm{d}\bar{l} = \frac{\mathrm{d}\bar{x}_l}{\mathrm{d}\lambda}\mathrm{d}\lambda, \tag{2.30}$$

so that (2.29) reduces to

$$\Gamma = \int_L \bar{v}(\bar{x}, t) \cdot \mathrm{d}\bar{l} = \int_{\lambda_1}^{\lambda_2} \bar{v}(\bar{x}_l(\lambda), t) \cdot \frac{\mathrm{d}\bar{x}_l}{\mathrm{d}\lambda}\mathrm{d}\lambda. \tag{2.31}$$

If one considers a closed curve, then according to Stokes theorem

$$\Gamma = \oint_L \bar{v} \cdot \mathrm{d}\bar{l} = \int_\Sigma (\nabla \wedge \bar{v}) \cdot \bar{n} \mathrm{d}\sigma, \tag{2.32}$$

where Σ represents any surface bounded by L and $\mathrm{d}\sigma$ is a surface differential element with normal unit vector $\bar{n}$, defined for the surface to have positive orientation according to the so-called right-hand rule; see Fig. 2.3. For the last equation to hold, the velocity field $\bar{v}(\bar{x}, t)$ must be continuous and the surface Σ must be completely contained within the fluid domain. The vector $\bar{\omega} = \nabla \wedge \bar{v}$ is called **vorticity**.

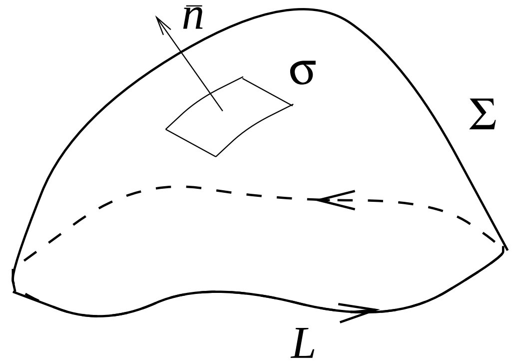

Figure 2.3: Application of Stokes theorem for the computation of the circulation.

The circulation along a closed curve limiting an infinitesimally small surface $\mathrm{d}\sigma$ contained in a plane normal to $\bar{n}$ reduces to

$$\oint_L \bar{v} \cdot \mathrm{d}\bar{l} = (\nabla \wedge \bar{v}) \cdot \bar{n} \mathrm{d}\sigma. \tag{2.33}$$

Hence, $(\nabla \wedge \bar{v})_n = (\nabla \wedge \bar{v}) \cdot \bar{n}$ turns out to be the circulation per unit surface around a curve normal to $\bar{n}$. At a given point, the circulation will be maximum when the curve is contained in a plane normal to $\nabla \wedge \bar{v}$.

Irrotational flow and velocity potential

The fluid motion is said to be irrotational when the vorticity is identically zero everywhere, i.e., $\bar{\omega} = \nabla \wedge \bar{v} = 0$. In vector calculus we have seen that for any irrotational vector field one can define a scalar function, called potential function, such that

$$\bar{v} = \nabla \varphi. \tag{2.34}$$

The potential is defined by (2.34), except for an integration constant, whose value, a function of time, can be selected arbitrarily. Clearly, the introduction of the velocity potential will simplify notably the description of irrotational motions, because we are effectively replacing the computation of a three-component vector field (the velocity $\bar{v}$) by that of a scalar field (the potential φ).

If the circulation along any closed curve is zero, the flow is necessarily irrotational. The demonstration begins by considering an infinitesimally small closed curve around a given point. According to (2.33), the circulation around the curve will be zero, regardless of the orientation $\bar{n}$, if

$\nabla \wedge \bar{v} = 0$. The opposite is not necessarily true, that is, even though $\nabla \wedge \bar{v} = 0$ everywhere in the flow field, the circulation around a closed curve might not be zero. This is so because the validity of (2.32) is restricted to surfaces Σ that are completely contained in the fluid domain. Therefore, for an irrotational field, one can claim that the circulation around a given closed curve is zero if and only if we can find a surface bounded by the curve that is completely contained in the flow field, that is, when the curve is continuously reducible to a point (the fluid domain is simply connected). This feature of irrotational motion arises in the study of flows over airfoils, relevant to the design of wings, compressor and turbine blades, etc. In the two-dimensional flow field that appears, the circulation around a closed curve circling the airfoil - which is non-reducible to a point - is non-zero, and turns out to be linearly proportional to the lift force provided by the airfoil.

Convective flux

Mass, momentum and energy are transported by the moving fluid. To quantify in general this transport rate, we introduce $\phi(\bar{x}, t)$ to represent a given fluid property expressed per unit volume, with particular cases of interest being the mass, momentum and energy per unit volume ($\phi = \rho$, $\phi = \rho\bar{v}$ and $\phi = \rho(e + |\bar{v}|^2/2)$). With this definition, the total amount of a given quantity existing within a given volume V is given by $\int_V \phi \mathrm{d}V$ (e.g., the total amount of mass contained in a volume is $\int_V \rho \mathrm{d}V$).
Let us now consider the fixed surface Σ_o shown in the figure. The fluid will cross the surface in its motion, transporting across mass, momentum and energy. The volume of fluid crossing in a

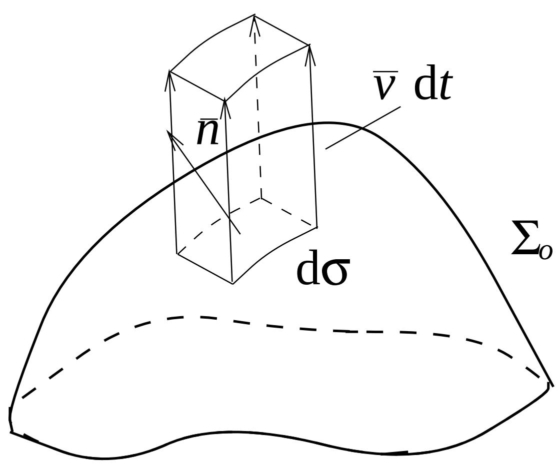

Figure 2.4: Convective flux.

time dt through the differential surface element dσ oriented perpendicular to $\bar{n}$ is that contained in the parallelepiped of base dσ and side $\bar{v}$dt shown in the figure, which can be computed as $\bar{v} \cdot \bar{n} \mathrm{d}\sigma \mathrm{d}t$. The amount of mass, momentum and energy crossing with the fluid is therefore given by $\phi\bar{v} \cdot \bar{n} \mathrm{d}\sigma \mathrm{d}t$, and the total convective flux across the surface (mass, momentum or energy that crosses the surface per unit time) is obtained by adding the contribution of the different surface elements and dividing the result by the unit time dt to give

$$\int_{\Sigma_o} \phi \bar{v} \cdot \bar{n} \mathrm{d}\sigma. \tag{2.35}$$

Note that, with $\phi = 1$, the above integral serves to compute the volume of fluid that crosses Σ_o per unit time, the **volume flux**

$$Q = \int_{\Sigma_o} \bar{v} \cdot \bar{n} \mathrm{d}\sigma. \tag{2.36}$$

When $\phi = \bar{v}$, we find inside the integral (2.35) the so-called momentum flux tensor $\rho\bar{v}\bar{v}$.
If Σ_o is a closed surface (and $\phi\bar{v}$ is a continuous function), Gauss formula (2.9) enables the convective flux to be written in the form

$$\int_{\Sigma_o} \phi \bar{v} \cdot \bar{n} \mathrm{d}\sigma = \int_{V_o} \nabla \cdot (\phi \bar{v}) \mathrm{d}V, \tag{2.37}$$

where V_o is the volume bounded by Σ_o. If we now consider an infinitesimally small volume, it then follows that $\nabla \cdot (\phi\bar{v})$ is the rate at which mass, momentum or energy abandons the unit volume due to the fluid outflow. Similarly, $\nabla \cdot \bar{v}$ is the amount of volume that abandons the unit volume per unit time (the expansion rate). Note that, for a perfect liquid, whose density is strictly constant, the condition that the specific volume cannot change requires that the condition

$$\nabla \cdot \bar{v} = 0 \tag{2.38}$$

be satisfied at each point.
The concept of convective flux can be extended to a moving surface $\Sigma_c(t)$ whose points are moving with velocity $\bar{v}_c(\bar{x}_c, t)$ by replacing the fluid velocity $\bar{v}$ with the relative velocity $\bar{v} - \bar{v}_c$ when computing the flux to give

$$\int_{\Sigma_c(t)} \phi(\bar{v} - \bar{v}_c) \cdot \bar{n} \mathrm{d}\sigma. \tag{2.39}$$

The stream function

The description of flows that are either planar or axisymmetric can be simplified when the velocity field is *solenoidal* (i.e., satisfies $\nabla \cdot \bar{v} = 0$), as occurs in constant-density fluids (see the discussion leading to (2.63)). To present the development, we restrict our attention to the case of a planar flow described in a cartesian $x - y$ coordinate system. Extensions to steady gas flow, for which the continuity equation becomes $\nabla \cdot (\rho\bar{v}) = 0$, and also to polar coordinates and axisymmetric flows can be performed, but will not be considered further in this introductory section.
For a planar flow $\nabla \cdot \bar{v} = 0$ reduces to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{2.40}$$

To solve the problem, it is convenient to introduce a scalar function ψ, called **stream function** defined from the two equations

$$v_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x}, \tag{2.41}$$

which determine ψ upon integration, except for an arbitrary additive constant. Straightforward substitution of (2.41) into (2.40) reveals that the velocity field deriving from a stream function satisfies automatically the condition of zero expansion rate. An important property of the stream function is that the isolines ψ = constant are streamlines, as can be seen by noticing that along an isoline

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial y}\mathrm{d}y = -v_y \mathrm{d}x + v_x \mathrm{d}y = 0. \qquad (2.42)$$

The last of these equations can be written as

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y}, \qquad (2.43)$$

corresponding to the planar counterpart of (2.21), thereby demonstrating that isolines and streamlines coincide.

Another useful property of ψ is that the difference $\psi_2 - \psi_1$ between the value of the stream function corresponding to two different streamlines equals the volumetric flux (2.36) for the planar stream tube defined by the two streamlines. Given a curve with end points on each one

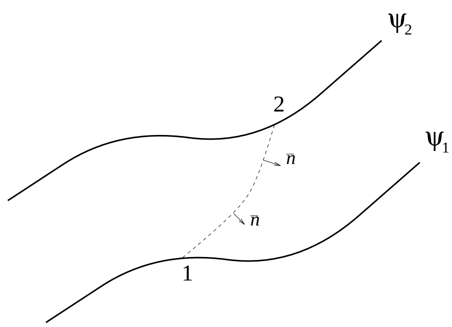

Figure 2.5: Volumetric flux between two streamlines.

of the streamlines, as shown in the figure, the value of $\psi_2 - \psi_1$ is given by the integral

$$\psi_2 - \psi_1 = \int_1^2 \mathrm{d}\psi = \int_1^2 (-v_y \mathrm{d}x + v_x \mathrm{d}y), \qquad (2.44)$$

as can be seen from (2.42). Bearing in mind that $(\mathrm{d}y, -\mathrm{d}x) = \bar{n}\mathrm{d}l$, where $\bar{n}$ is the unit vector normal to the integration curve, the previous equation can finally be rewritten in the form

$$\psi_2 - \psi_1 = \int_1^2 \bar{v} \cdot \bar{n} \mathrm{d}l, \qquad (2.45)$$

where the integral on the right-hand side is the volumetric flux between the two stream surfaces considered (per unit length perpendicular to the plane)

Relative motion near a point

As previously mentioned, the force exerted by a portion of fluid on an adjacent portion of fluid is proportional to the rate at which the fluid is being deformed. To determine that rate, let us consider the motion of a differential element of fluid line $\mathrm{d}\bar{x}$, whose ends are initially located at $\bar{x}$ and $\bar{x}+\mathrm{d}\bar{x}$. The velocities of these two ends differ by a small amount $\mathrm{d}\bar{v}$, which can be expressed in the first approximation as

$$\mathrm{d}\bar{v} = \mathrm{d}\bar{x}\cdot\nabla\bar{v} \tag{2.46}$$

in terms of the gradient of velocity $\nabla\bar{v}$. After an infinitesimally small time $\mathrm{d}t$, the two ends of the fluid line element have moved to occupy new positions given by $\bar{x}+\bar{v}\mathrm{d}t$ and $\bar{x}+\mathrm{d}\bar{x}+(\bar{v}+\mathrm{d}\bar{v})\mathrm{d}t$, respectively. Clearly, besides a translation $\bar{v}\mathrm{d}t$, the fluid element has undergone a distortion, with $\mathrm{d}\bar{x}$ becoming $\mathrm{d}\bar{x}+\mathrm{d}\bar{v}\mathrm{d}t$, as indicated in Fig. 2.6.

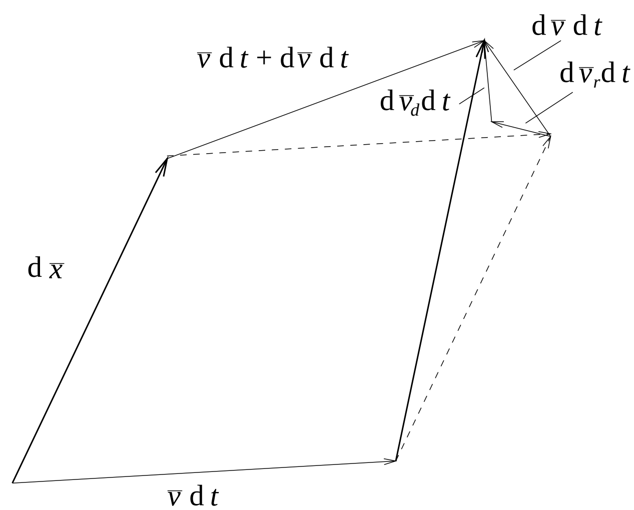

Figure 2.6: Motion of a differential element of fluid line.

The geometrical character of $\mathrm{d}\bar{v}\mathrm{d}t = \mathrm{d}\bar{x}\cdot\nabla\bar{v}\mathrm{d}t$ is best investigated by decomposing the velocity gradient into parts that are symmetric and anti-symmetric, according to

$$\nabla\bar{v} = \frac{1}{2}(\nabla\bar{v}+\nabla\bar{v}^{\mathrm{T}}) + \frac{1}{2}(\nabla\bar{v}-\nabla\bar{v}^{\mathrm{T}}) = \bar{\bar{\mathrm{T}}}_{\mathrm{d}} + \bar{\bar{\mathrm{T}}}_{\mathrm{r}}. \tag{2.47}$$

where the symmetric tensor $\bar{\bar{\mathrm{T}}}_{\mathrm{d}}$ is called the rate-of-strain tensor and the anti-symmetric tensor $\bar{\bar{\mathrm{T}}}_{\mathrm{r}}$ is called the rotation tensor. In cartesian coordinates, the two tensors $\bar{\bar{\mathrm{T}}}_{\mathrm{d}}$ and $\bar{\bar{\mathrm{T}}}_{\mathrm{r}}$ can be computed according to

$$\bar{\bar{\mathrm{T}}}_{\mathrm{d}} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1}+\frac{\partial v_1}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2}+\frac{\partial v_2}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1}+\frac{\partial v_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2}+\frac{\partial v_2}{\partial x_3}\right) & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \tag{2.48}$$

and

$$\bar{\bar{\mathrm{T}}}_{\mathrm{r}} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1}-\frac{\partial v_1}{\partial x_3}\right) \\ -\frac{1}{2}\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right) \\ -\frac{1}{2}\left(\frac{\partial v_3}{\partial x_1}-\frac{\partial v_1}{\partial x_3}\right) & -\frac{1}{2}\left(\frac{\partial v_3}{\partial x_2}-\frac{\partial v_2}{\partial x_3}\right) & 0 \end{bmatrix} \tag{2.49}$$

This last expression can be rewritten in the form

$$\bar{\bar{\mathrm{T}}}_{\mathrm{r}} = \frac{1}{2}\begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}, \tag{2.50}$$

in terms of the three components ω_1, ω_2 and ω_3 of the vorticity vector $\bar{\omega} = \nabla \wedge \bar{v}$.
Introducing (2.47) into (2.46) yields

$$\mathrm{d}\bar{v} = \mathrm{d}\bar{x} \cdot \bar{\bar{\mathrm{T}}}_{\mathrm{d}} + \mathrm{d}\bar{x} \cdot \bar{\bar{\mathrm{T}}}_{\mathrm{r}} = \mathrm{d}\bar{v}_d + \mathrm{d}\bar{v}_r. \tag{2.51}$$

The contribution of the anti-symmetric tensor $\bar{\bar{\mathrm{T}}}_{\mathrm{r}}$ corresponds to a rotation of the fluid element $\mathrm{d}\bar{x}$ with angular velocity $(\nabla \wedge \bar{v})/2$, as can be seen by writing $\mathrm{d}\bar{v}_r$ in the form

$$\mathrm{d}\bar{v}_r = \mathrm{d}\bar{x} \cdot \bar{\bar{\mathrm{T}}}_{\mathrm{r}} = \frac{1}{2}(\nabla \wedge \bar{v}) \wedge \mathrm{d}\bar{x} = \frac{1}{2}\,\bar{\omega} \wedge \mathrm{d}\bar{x}. \tag{2.52}$$

Thus, if $\bar{\bar{\mathrm{T}}}_{\mathrm{d}}$ were identically zero, the motion of the fluid would be exactly that of a rigid body, that is, exclusively translation and rotation.
On the other hand, the symmetry of the rate-of-strain tensor $\bar{\bar{\mathrm{T}}}_{\mathrm{d}}$ enables $\mathrm{d}\bar{v}_d = \mathrm{d}\bar{x} \cdot \bar{\bar{\mathrm{T}}}_{\mathrm{d}}$ to be written in the form

$$\mathrm{d}\bar{v}_d = \bar{\bar{\mathrm{T}}}_{\mathrm{d}} \cdot \bar{n}\,\mathrm{d}s, \tag{2.53}$$

where the differential fluid element has been expressed as $\mathrm{d}\bar{x} = \bar{n}\mathrm{d}s$ in terms of its differential length $\mathrm{d}s$ and the unit vector $\bar{n}$. In general, the strain rate is not aligned with $\bar{n}$, implying that the element $\mathrm{d}\bar{x}$ undergoes both extension and shear strain. The magnitude of the extension rate is given by the scalar product $\bar{n} \cdot \bar{\bar{\mathrm{T}}}_{\mathrm{d}} \cdot \bar{n}\,\mathrm{d}s$, so that $\bar{n} \cdot \bar{\bar{\mathrm{T}}}_{\mathrm{d}} \cdot \bar{n}$ represents the extension rate per unit length along the direction defined by $\bar{n}$. On the other hand, the contribution of $\mathrm{d}\bar{v}_d$ to the shear deformation is obtained by appropriately subtracting from the total strain rate the extension rate according to $[\bar{\bar{\mathrm{T}}}_{\mathrm{d}} \cdot \bar{n} - (\bar{n} \cdot \bar{\bar{\mathrm{T}}}_{\mathrm{d}} \cdot \bar{n})\,\bar{n}]\mathrm{d}s$.
There are three privileged directions of space, called principal directions of strain, along which the strain reduces to an extension, without any shear strain (the resulting vector $\mathrm{d}\bar{v}_d$ is parallel to $\bar{n}$). These principal directions $\bar{n}_1$, $\bar{n}_2$ and $\bar{n}_3$, and their corresponding strain rates, λ_1, λ_2 and λ_3, are computed by solving the eigenvalue problem

$$\bar{\bar{\mathrm{T}}}_{\mathrm{d}} \cdot \bar{n} = \lambda\,\bar{n}. \tag{2.54}$$

In particular, the characteristic equation that determines λ_i,

$$|\bar{\bar{\mathrm{T}}}_{\mathrm{d}} - \lambda\,\bar{\bar{\mathrm{I}}}| = 0 \tag{2.55}$$

follows from imposing the existence of nontrivial solutions. The symmetry of the tensor $\bar{\bar{\mathrm{T}}}_{\mathrm{d}}$ guarantees the existence of three real roots for this equation, associated with three mutually perpendicular principal directions. Obviously, in the local reference frame defined by the three directions $\bar{n}_i$ the associated rate-of-strain tensor becomes diagonal, with diagonal components λ_1, λ_2 and λ_3.

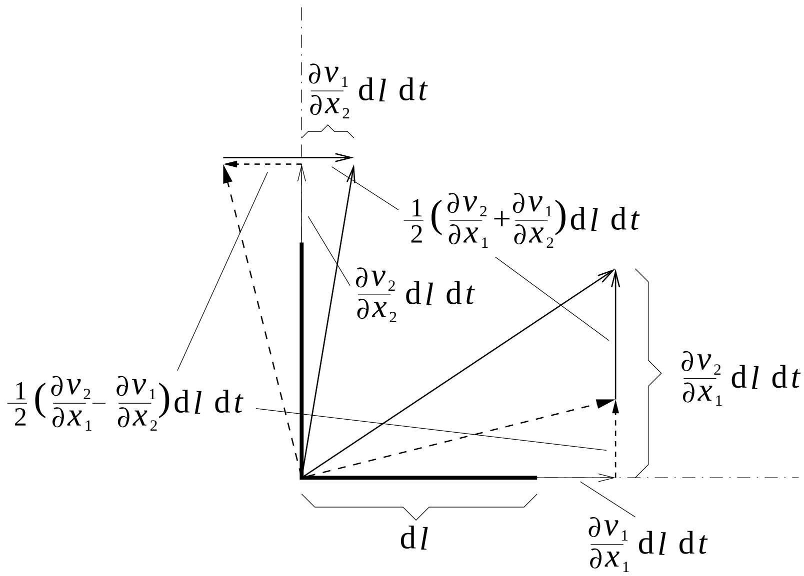

Figure 2.7: Deformation and rotation of an infinitesimally small fluid element of square shape.

Deformation of a square fluid element

Let us now consider the evolution of a square fluid element. The two sides, of initial length dl, evolve after a time dt from d$\bar{l}$ to d$\bar{l}$ + d$\bar{l} \cdot \nabla\bar{v}dt$, so that

$$\mathrm{d}\bar{l}\cdot\nabla\bar{v} = \mathrm{d}l(1,0)\begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} \end{bmatrix} = \mathrm{d}l\left(\frac{\partial v_1}{\partial x_1}, \frac{\partial v_2}{\partial x_1}\right) \tag{2.56}$$

and

$$\mathrm{d}\bar{l}\cdot\nabla\bar{v} = \mathrm{d}l(0,1)\begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} \end{bmatrix} = \mathrm{d}l\left(\frac{\partial v_1}{\partial x_2}, \frac{\partial v_2}{\partial x_2}\right) \tag{2.57}$$

The observation of Fig. 2.7 helps to reveal the physical meaning of each of the terms appearing in the tensors

$$\bar{\bar{\mathrm{T}}}_\mathrm{d} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\right) & \frac{\partial v_2}{\partial x_2} \end{bmatrix} \tag{2.58}$$

and

$$\bar{\bar{\mathrm{T}}}_\mathrm{r} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\right) & 0 \end{bmatrix} \tag{2.59}$$

Recalling that the decomposition $\mathrm{d}\bar{l}\cdot\nabla\bar{v}\mathrm{d}t = \mathrm{d}\bar{v}_d\mathrm{d}t + \mathrm{d}\bar{v}_r\mathrm{d}t$ applies, with $\mathrm{d}\bar{v}_d = \mathrm{d}\bar{l}\cdot\bar{\bar{\mathrm{T}}}_\mathrm{d}$ and

$d\bar{v}_r = d\bar{l} \cdot \bar{\bar{T}}_r$, one obtains for the horizontal side $dl(1, 0)$

$$d\bar{v}_d = dl \left[\frac{\partial v_1}{\partial x_1}, \frac{1}{2} \left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) \right] \text{ and } d\bar{v}_r = dl \left[0, \frac{1}{2} \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \right], \tag{2.60}$$

whereas for the vertical side

$$d\bar{v}_d = dl \left[\frac{1}{2} \left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right), \frac{\partial v_2}{\partial x_2} \right] \text{ and } d\bar{v}_r = dl \left[-\frac{1}{2} \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right), 0 \right]. \tag{2.61}$$

According to the figure, the elements along the diagonal of the rate-of-strain tensor represent the rate of extension per unit length in the directions of the three axes. Besides, the sides have undergone rotations of angles $dt(\partial v_2/\partial x_1)$ and $-dt(\partial v_1/\partial x_2)$, so that the average angular velocity for the square element in its plane is given by $\frac{1}{2}(\partial v_2/\partial x_1 - \partial v_1/\partial x_2)$. Therefore, the off-diagonal elements of the anti-symmetric tensor $\bar{\bar{T}}_r$ represent the three components of the angular velocity of the fluid element. On the other hand, the angle formed by the fluid-element sides, initially perpendicular, have decreased an amount $(\partial v_2/\partial x_1 + \partial v_1/\partial x_2)dt$, indicating that $(\partial v_2/\partial x_1 + \partial v_1/\partial x_2)$ is the shear strain rate (the rate at which the angle formed by the directions 1 and 2 is decreasing). In other words, the off-diagonal components in $\bar{\bar{T}}_d$ correspond to half of the shear strain rate perpendicular to the three axis of the reference frame.

Deformation of a cubic fluid element

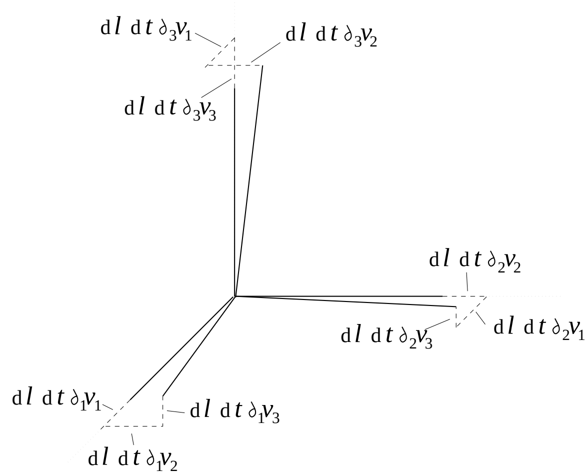

Figure 2.8: Deformation and rotation of a cubic fluid element of infinitesimally small size.

The above figure considers the evolution of an infinitesimally small fluid element of cubic shape, with the simplified notation $\partial_i v_j = \partial v_j/\partial x_i$ adopted for convenience. The discussion given above for the square shape also applies here. In this case, the distortion of the fluid element associated with the straining motion produces a variation of volume from its initial value dl^3. After a differential time dt, the new volume is given by

$$dl^3 \begin{vmatrix} 1 + \partial_1 v_1 dt & \partial_1 v_2 dt & \partial_1 v_3 dt \\ \partial_2 v_1 dt & 1 + \partial_2 v_2 dt & \partial_2 v_3 dt \\ \partial_3 v_1 dt & \partial_3 v_2 dt & 1 + \partial_3 v_3 dt \end{vmatrix} \simeq dl^3 + dl^3 dt(\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3), \tag{2.62}$$

after small terms of order $\mathrm{d}t^2$ and $\mathrm{d}t^3$ are neglected. The previous equation reveals that the velocity divergence $\nabla \cdot \bar{v} = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$ represents physically the rate at which the volume of the fluid element is changing per unit volume. This quantity $\nabla \cdot \bar{v}$, equal to the trace of the tensors $\nabla \bar{v}$ and $\bar{\bar{\mathrm{T}}}_{\mathrm{d}}$, is called **rate of expansion** or **dilatation rate**. For a perfect liquid, for which the volume of the fluid element does not change since the density is constant, the velocity field must satisfy

$$\nabla \cdot \bar{v} = 0, \tag{2.63}$$

a result that was anticipated earlier. Clearly, for the dilatation rate to be zero, the linear extension rates $\partial_1 v_1$, $\partial_2 v_2$ and $\partial_3 v_3$ cannot all three have the same sign. In other words, for the volume of an incompressible fluid particle to remain constant, a positive extension rate appears in one or two directions, with a compensating compression rate appearing in the other one (or two) directions to give a zero net expansion rate.