

Chapter 3

Conservation laws

Reynolds transport theorem

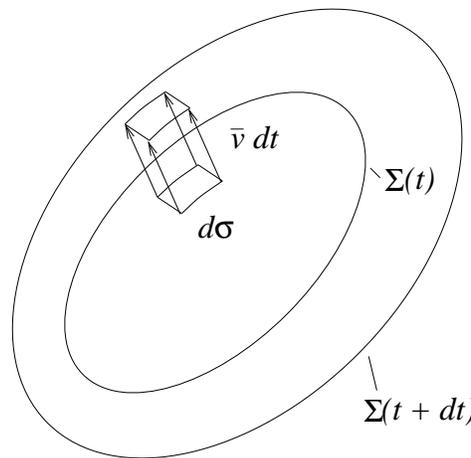


Figure 3.1: Evolution of a fluid volume.

To write the conservation equations, it is of interest to determine the variation with time of the amount of mass, momentum or energy contained in a fluid volume $V_f(t)$ limited externally by a closed fluid surface $\Sigma_f(t)$

$$\frac{d}{dt} \left[\int_{V_f(t)} \phi(\bar{x}, t) dV \right]; \quad \phi = \rho, \rho \bar{v}, \rho(e + |\bar{v}|^2/2). \quad (3.1)$$

Since both the integrand $\phi(\bar{x}, t)$ and the integration domain $V_f(t)$ vary with time, one can expect two different contributions associated, respectively, with the unsteadiness of the flow field and with the motion of the fluid, just as we saw earlier in connection with the material derivative. To quantify the two contributions, it is convenient to consider the evolution of the fluid volume between two successive instants of time t and $t + dt$, as sketched in Fig. 3.1.

Equation (3.1) can be written using the definition of limit

$$\frac{d}{dt} \left[\int_{V_f(t)} \phi(\bar{x}, t) dV \right] = \lim_{dt \rightarrow 0} \frac{\int_{V_f(t+dt)} \phi(\bar{x}, t+dt) dV - \int_{V_f(t)} \phi(\bar{x}, t) dV}{dt}, \quad (3.2)$$

which can also be split into two separate contributions according to

$$\lim_{dt \rightarrow 0} \frac{\int_{V_f(t)} [\phi(\bar{x}, t+dt) - \phi(\bar{x}, t)] dV}{dt} + \lim_{dt \rightarrow 0} \frac{\int_{V_f(t+dt)-V_f(t)} \phi(\bar{x}, t+dt) dV}{dt}. \quad (3.3)$$

In evaluating the first term we use the expansion $\phi(\bar{x}, t+dt) - \phi(\bar{x}, t) = (\partial\phi/\partial t)dt + O(dt^2)$ to give in the first approximation

$$\lim_{dt \rightarrow 0} \frac{\int_{V_f(t)} [\phi(\bar{x}, t+dt) - \phi(\bar{x}, t)] dV}{dt} = \int_{V_f(t)} \frac{\partial\phi(\bar{x}, t)}{\partial t} dV, \quad (3.4)$$

while the second integral can be evaluated as the convective flux crossing the surface $\Sigma_f(t)$ during the time interval dt to give

$$\lim_{dt \rightarrow 0} \frac{\int_{V_f(t+dt)-V_f(t)} \phi(\bar{x}, t+dt) dV}{dt} = \lim_{dt \rightarrow 0} \frac{\int_{\Sigma_f(t)} \phi \bar{v} \cdot \bar{n} d\sigma dt}{dt} = \int_{\Sigma_f(t)} \phi \bar{v} \cdot \bar{n} d\sigma, \quad (3.5)$$

finally yielding

$$\frac{d}{dt} \left[\int_{V_f(t)} \phi(\bar{x}, t) dV \right] = \int_{V_f(t)} \frac{\partial\phi(\bar{x}, t)}{\partial t} dV + \int_{\Sigma_f(t)} \phi(\bar{x}, t) \bar{v}(\bar{x}, t) \cdot \bar{n} d\sigma. \quad (3.6)$$

The two terms in (3.6) correspond to the previously anticipated contributions arising from flow unsteadiness and from fluid motion, respectively.

The solution of a given problem involves in general the variation with time of fluid properties in a finite region $V_c(t)$ that, depending on the problem, can be fixed or variable in time. That region is called a **control volume**, which is limited externally by a control surface $\Sigma_c(t)$, a closed surface whose points move with velocity $\bar{v}_c(\bar{x}_c, t)$. Since the conservation laws of fluid mechanics apply to fluid volumes, and not to control volumes, it is of interest to relate the time derivatives extended to $V_c(t)$ to those extended to a control volume $V_f(t)$ occupying the same space at the instant considered. For $V_c(t)$, the reasoning that led before to (3.6) allows us to write

$$\frac{d}{dt} \left[\int_{V_c(t)} \phi(\bar{x}, t) dV \right] = \int_{V_c(t)} \frac{\partial\phi(\bar{x}, t)}{\partial t} dV + \int_{\Sigma_c(t)} \phi(\bar{x}, t) \bar{v}_c(\bar{x}, t) \cdot \bar{n} d\sigma. \quad (3.7)$$

Bearing in mind that we chose the control volume $V_c(t)$ and its bounding surface $\Sigma_c(t)$ to coincide exactly with $V_f(t)$ and $\Sigma_f(t)$ at the instant considered, the integration domains that appear in (3.6) and (3.7) are identical, and straightforward subtraction of both expressions provides

$$\frac{d}{dt} \left[\int_{V_f(t)} \phi(\bar{x}, t) dV \right] = \frac{d}{dt} \left[\int_{V_c(t)} \phi(\bar{x}, t) dV \right] + \int_{\Sigma_c(t)} \phi(\bar{x}, t) (\bar{v} - \bar{v}_c) \cdot \bar{n} d\sigma. \quad (3.8)$$

This equation, to be used below when deriving the conservation equations for an arbitrary control volume, indicates that the time variation of the given flow property (mass, momentum or energy) in a fluid volume $V_f(t)$ equals the time variation in a control volume coinciding with $V_f(t)$ plus the convective flux across the bounding surface $\Sigma_c(t)$ (see (2.39)).

Mass conservation

The first conservation law of fluid mechanics states that the mass of a given fluid volume remains constant,

$$\frac{d}{dt} \left[\int_{V_f(t)} \rho dV \right] = 0, \quad (3.9)$$

which can be expressed with use made of (3.8) in the form

$$\frac{d}{dt} \left[\int_{V_c(t)} \rho dV \right] + \int_{\Sigma_c(t)} \rho(\bar{v} - \bar{v}_c) \cdot \bar{n} d\sigma = 0. \quad (3.10)$$

The physical meaning underlying the above equation becomes clear by writing the second integral on the right-hand side of the equation to give

$$\frac{d}{dt} \left[\int_{V_c(t)} \rho dV \right] = - \int_{\Sigma_c(t)} \rho(\bar{v} - \bar{v}_c) \cdot \bar{n} d\sigma = 0,$$

which states that the rate of increment of the mass contained in a control volume equals the mass flux entering across its bounding surface, $-\int_{\Sigma_c(t)} \rho(\bar{v} - \bar{v}_c) \cdot \bar{n} d\sigma$. When the control volume is fixed in space, the above equation reduces to

$$\frac{d}{dt} \left[\int_{V_o} \rho dV \right] + \int_{\Sigma_o} \rho \bar{v} \cdot \bar{n} d\sigma = 0. \quad (3.11)$$

Momentum conservation

Volume and surface forces

The forces acting on a fluid (or on a solid) can be classified in two different categories: long-range forces, also called volume or mass forces, and short-range forces, also called surface forces. The former, including for instance gravity and inertial forces, decay slowly over distances much larger than the intermolecular distance d and therefore act over the whole fluid volume (for instance, gravity decays over distances that are of the order of the Earth radius). Short-range forces, on the other hand, have a direct molecular origin and decay rapidly with distance, their range of influence being of the order of the intermolecular distance d .

The mass force acting on a fluid particle of volume dV can be expressed in the form

$$\rho \bar{f}_m(\bar{x}, t) dV, \quad (3.12)$$

where $\bar{f}_m(\bar{x}, t)$ is the volume force per unit mass, which equals $\bar{f}_m(\bar{x}, t) = \bar{g}$ when gravity is the only mass force acting on the system. If a non-inertial reference frame is selected in the description, then inertial forces must be also considered to give

$$\bar{f}_m(\bar{x}, t) = \bar{g} - \bar{a}_o - \frac{d\bar{\Omega}}{dt} \wedge \bar{x} - \bar{\Omega} \wedge (\bar{\Omega} \wedge \bar{x}) - 2\bar{\Omega} \wedge \bar{v}, \quad (3.13)$$

with \bar{a}_o and $\bar{\Omega}$ representing the acceleration and angular velocity of the non-inertial reference frame. To write (3.12) we have neglected the variation of \bar{f}_m within the fluid particle, an accurate

approximation given the long-range nature of these forces. Some mass forces are conservative and derive from a potential according to $\bar{f}_m = -\nabla U$, so that, for instance,

$$\bar{g} - \bar{a}_o - \bar{\Omega} \wedge (\bar{\Omega} \wedge \bar{x}) = -\nabla[-\bar{g} \cdot \bar{x} + \bar{a}_o \cdot \bar{x} - (\bar{\Omega} \wedge \bar{x}) \cdot (\bar{\Omega} \wedge \bar{x})/2]. \quad (3.14)$$

On the other hand, short-range forces only reach a short distance of the order of d . For a gas, they are directly related with the flux of momentum emerging due to the random molecular motion. The molecules crossing a surface separating two adjacent portions of fluid transport their momentum and energy. This microscopic transport effect produces a surface force at the macroscopic level, and also gives rise to heat conduction. If the fluid is a liquid, then there exist additional contributions due to the forces between molecules located on both sides of the surface. Because of their short-range nature, the force acting on a fluid parcel is proportional to its surface, so that given a surface element $d\sigma$ with orientation \bar{n} separating two fluid elements the resulting force is given by

$$\bar{f}_n(\bar{n}, \bar{x}, t)d\sigma, \quad (3.15)$$

where \bar{f}_n is the force per unit surface (**stress**), a function of \bar{x} , t and, also, of the orientation \bar{n} . In the convention adopted, \bar{f}_n is the stress exerted by the fluid on the side of the surface element to which \bar{n} points on the fluid located on the opposite side.

Stress tensor

The dependence of the stress \bar{f}_n on \bar{n} can be shown to reduce to a linear relationship, given below in (3.19), involving a second-order symmetric tensor $\bar{\tau}$ called the stress tensor. To derive this equation, one applies Newton's second law to the fluid element shown in Fig. 3.2, whose initial shape is that of a tetrahedron of sides dx_1 , dx_2 and dx_3 and volume $dV = (1/6)dx_1dx_2dx_3$. Both

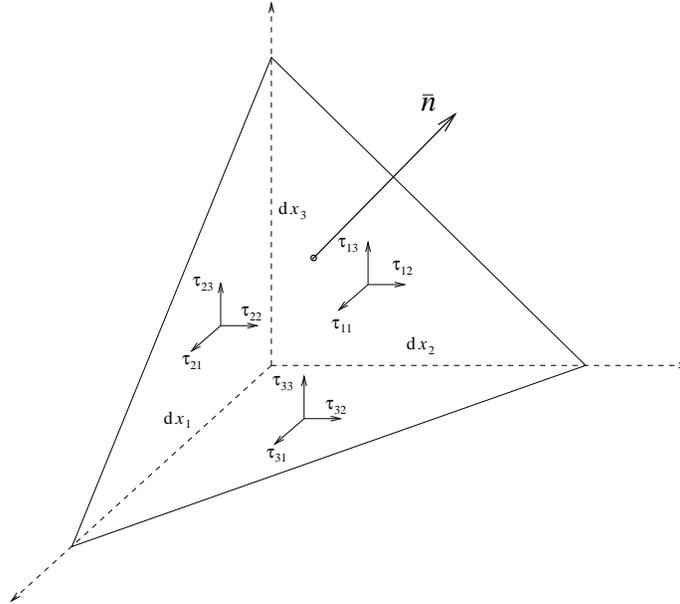


Figure 3.2: Stresses on a tetrahedral fluid element.

the inertial term (mass ρdV times acceleration $D\bar{v}/Dt$) and the resultant mass force $\rho dV \bar{f}_m$ are

proportional to the volume of the element, while the surface forces acting on the four sides of the tetrahedron are proportional to their surfaces. Hence, for vanishingly small values of $dx \ll 1$, the momentum equation reduces to a balance between the surface forces. In expressing the equilibrium balance, we shall use $\bar{f}_i = (\tau_{i1}, \tau_{i2}, \tau_{i3})$ ($i = 1, 2, 3$) to denote the stress (force per unit surface) acting on a surface normal to the unit vector \bar{e}_i of the reference frame. Consequently, with the sign convention adopted, the force acting on the tetrahedron-shaped fluid element through the face contained in a plane normal to \bar{e}_i is given by $-dA_i \bar{f}_i$, whereas that applied through the oblique face is $dA \bar{f}_n$, with $\bar{n} = (n_1, n_2, n_3)$ denoting the unit vector normal to the oblique face. With this notation, the equilibrium equation reduces to

$$dA \bar{f}_n - dA_1 \bar{f}_1 - dA_2 \bar{f}_2 - dA_3 \bar{f}_3 = 0. \quad (3.16)$$

Using the geometrical identity $dA_i = n_i dA$, where dA denotes the surface area of the oblique face, allows us to rewrite the previous equation in the form

$$\bar{f}_n = n_1 \bar{f}_1 + n_2 \bar{f}_2 + n_3 \bar{f}_3 = \bar{n} \cdot \bar{\tau}, \quad (3.17)$$

where

$$\bar{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \quad (3.18)$$

is the so-called stress tensor. Equation (3.17) indicates that the stress acting on a surface of orientation \bar{n} can be expressed as a function of the nine different components of the tensor $\bar{\tau}$, which are in principle a function of \bar{x} and t .

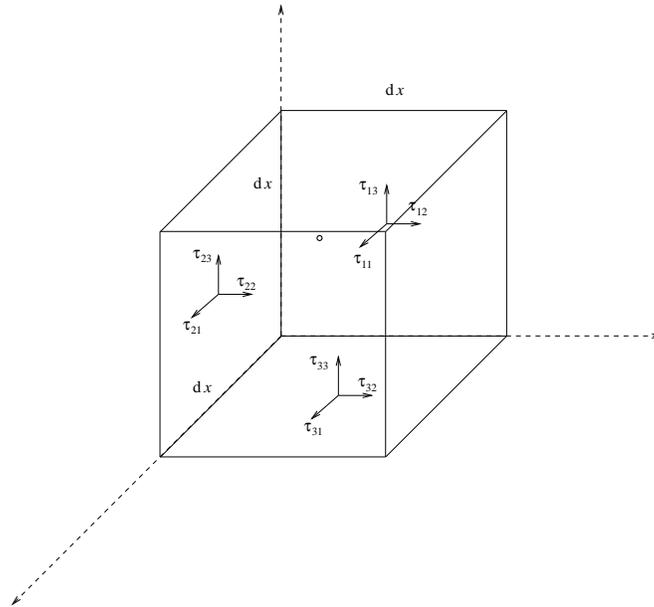


Figure 3.3: Stresses on a cubic element.

The result is even simpler, in that the stress tensor is actually symmetrical, and therefore contains only six different components. To show the symmetry, one applies the conservation of angular

momentum to the cubic fluid element of side dx shown in Fig. 3.3. In the limit $dx \rightarrow 0$, the surface forces and their associated momenta with respect to any point must be in equilibrium. For instance, the equation imposing a zero torque in the vertical direction reads $dx^3(\tau_{21} - \tau_{12}) = 0$, indicating that $\tau_{12} = \tau_{21}$, and similar arguments in the other two directions yield $\tau_{23} = \tau_{32}$ and $\tau_{13} = \tau_{31}$, thereby demonstrating the aforementioned symmetry. The stress $\bar{f}_n(\bar{n}, \bar{x}, t)$ can therefore be expressed in the form

$$\bar{f}_n = \bar{\tau} \cdot \bar{n} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \tau_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \tau_{33} \end{bmatrix} \cdot \bar{n}, \quad (3.19)$$

where the normal stresses τ_{11} , τ_{22} and τ_{33} as well as the shear (tangential) stresses τ_{12} , τ_{13} and τ_{23} are functions of the position and time. Their dependences on the strain rate and on the local thermodynamic state of the fluid are to be studied in the next section for the case of newtonian fluids.

The stress $\bar{\tau} \cdot \bar{n}$ can be split into its normal component $(\bar{n} \cdot \bar{\tau} \cdot \bar{n})\bar{n}$ and its tangential component $\bar{\tau} \cdot \bar{n} - (\bar{n} \cdot \bar{\tau} \cdot \bar{n})\bar{n}$, the latter called shear stress. There exist three directions, called the principal directions of stress, for which the resulting shear stress is identically zero, i.e., the only stress component is normal, a condition expressed mathematically by the eigenvalue problem

$$\bar{\tau} \cdot \bar{n} = \lambda \bar{n}. \quad (3.20)$$

The symmetry of the stress tensor $\bar{\tau}$ guarantees the existence of three real roots of the associated characteristic equation

$$|\bar{\tau} - \lambda \bar{1}| = 0. \quad (3.21)$$

These roots are the normal stresses in the principal directions, which are of course mutually orthogonal.

The force acting on a given surface can be obtained by integrating (3.19) to give

$$\int_{\Sigma} \bar{f}_n d\sigma = \int_{\Sigma} \bar{\tau} \cdot \bar{n} d\sigma. \quad (3.22)$$

For a closed surface, Gauss formula (2.9) enables the previous equation to be written in the form

$$\int_{\Sigma} \bar{\tau} \cdot \bar{n} d\sigma = \int_V \nabla \cdot \bar{\tau} dV. \quad (3.23)$$

Evaluating the previous equation for a differential fluid element we conclude that $\nabla \cdot \bar{\tau}$ is the resultant surface force per unit volume, so that $(\nabla \cdot \bar{\tau})dV$ is the surface force acting on a fluid particle of volume dV . Noting that the mass of the fluid particle is given by ρdV , straightforward application of Newton's second law provides

$$\rho dV \frac{D\bar{v}}{Dt} = \nabla \cdot \bar{\tau} dV + \rho \bar{f}_m dV, \quad (3.24)$$

which incorporates also the effect of the mass forces possibly acting on the fluid. By eliminating dV one arrives at the differential form of the momentum equation

$$\rho \frac{D\bar{v}}{Dt} = \nabla \cdot \bar{\tau} + \rho \bar{f}_m, \quad (3.25)$$

to be derived in the next chapter.

Navier-Poisson equation

If the fluid is at rest in a given reference frame, the surface forces act always normal to the surface and can be in general written as

$$\bar{f}_n = -p\bar{n}, \quad (3.26)$$

where p is the pressure, a thermodynamic variable introduced above when writing the equations of state. Correspondingly, the stress tensor reduces in this case to

$$\bar{\tau} = -p\bar{\mathbb{I}}, \quad (3.27)$$

where $\bar{\mathbb{I}}$ represents the identity tensor.

Note that if $\bar{\mathbb{T}}_d = \nabla\bar{v} - \bar{\mathbb{T}}_r$ is zero in some reference frame, then it is also zero when a different reference frame is used, so that the fluid motion reduces to a local translation and rotation as a rigid body. In an appropriately selected local reference frame the fluid would be at rest, so that one may conclude that the result $\bar{\tau} = -p\bar{\mathbb{I}}$ applies in general whenever $\bar{\mathbb{T}}_d = 0$.

When the rate-of-strain tensor $\bar{\mathbb{T}}_d$ is non-zero (which occurs independently of the reference frame employed), besides the pressure forces there exist additional surface forces, called viscous forces, that can be expressed in terms of the viscous stress tensor $\bar{\tau}'$, defined such that

$$\bar{\tau} = -p\bar{\mathbb{I}} + \bar{\tau}'. \quad (3.28)$$

It must be pointed out that, up to this point, no hypotheses have been made about the nature of the fluid, other than the existence of local thermodynamic equilibrium. However, in order to relate the viscous stress tensor to the kinematic state of the flow at a particular point, it is necessary to make assumptions about the functional form of this relationship. Although it could be quite complex, for the so-called **newtonian fluids**, which include most of the fluids of engineering interest (e.g., air and water), it is observed that there exists a linear proportionality between the components τ'_{ij} of the viscous-stress tensor $\bar{\tau}'$ and the components γ_{kl} of the rate-of-strain tensor $\bar{\mathbb{T}}_d$. In principle, since both $\bar{\tau}'$ and $\bar{\mathbb{T}}_d$ are symmetrical, this proportionality condition leads to the six equations

$$\tau'_{ij} = \alpha_{ijkl}\gamma_{kl}, \quad (3.29)$$

involving the 36 proportionality constants α_{ijkl} .

Equations (3.29) simplify considerably for isotropic fluids, in that the number of proportionality constants reduces to only two. The demonstration begins by noting that the principal directions of strain and the principal directions of stress coincide for an isotropic fluid, and could be used as a local reference frame to express $\bar{\tau}'$ and $\bar{\mathbb{T}}_d$ in diagonal form. In the following, τ'_i and γ_i represent the stresses and strain rates in the principal directions. Evaluating (3.29) for τ'_1 and bearing in mind the isotropy of the fluid one obtains $\tau'_1 = \alpha\gamma_1 + \lambda\gamma_2 + \lambda\gamma_3$, where the coefficients α and λ relate the strain rate in a given direction with the viscous stresses that appear in that direction and in the two perpendicular ones, respectively. Analogous expressions, involving also α and λ would be obtained for τ'_2 and τ'_3 . The result can be expressed in the compact form $\tau'_i = (\alpha - \lambda)\gamma_i + \lambda\nabla \cdot \bar{v}$ or, alternatively,

$$\bar{\tau}' = 2\mu\bar{\mathbb{T}}_d + \lambda(\nabla \cdot \bar{v})\bar{\mathbb{I}}, \quad (3.30)$$

where the constants $\mu = (\alpha - \lambda)/2$ and λ are the first and second viscosity coefficients. This last expression remains valid when the principal directions are replaced by any other orthogonal

reference frame (remember that the expansion rate $\nabla \cdot \bar{v}$ is also invariant under the transformation). Following the most commonly used notation, the above expression is written in terms of the bulk viscosity coefficient $\mu_B = \lambda + \frac{2}{3}\mu$ to give

$$\bar{\tau}' = 2\mu\bar{\Gamma}_d + (\mu_B - \frac{2}{3}\mu)(\nabla \cdot \bar{v})\bar{\mathbb{I}}, \quad (3.31)$$

which further simplifies for an incompressible fluid, for which the second term vanishes. The expressions for $\nabla\bar{v}$ and $\nabla \cdot \bar{v}$ given previously can be used to express $\bar{\tau}'$ in terms of the derivatives of the different velocity components.

As can be seen, the viscous stresses depend on the strain rate, which appears explicitly in (3.31), and also on the local thermodynamic state, through the viscosity coefficients μ and μ_B . Besides these two coefficients, a related quantity, which becomes useful when comparing the molecular transport of mass, momentum and energy, is the so-called kinematic viscosity $\nu = \mu/\rho$. Characteristic values of ν for air and water at ambient pressure are, respectively, $\nu_a = 1.48 \times 10^{-5} \text{ m}^2/\text{s}$ and $\nu_w = 1.14 \times 10^{-6} \text{ m}^2/\text{s}$ at $T = 288 \text{ K}$, and $\nu_a = 2.24 \times 10^{-5} \text{ m}^2/\text{s}$ and $\nu_w = 0.31 \times 10^{-6} \text{ m}^2/\text{s}$ at $T = 368 \text{ K}$.

The existence of local thermodynamic equilibrium enables the dependence of μ and μ_B on the thermodynamic state of the fluid to be studied in terms of any two independent thermodynamic variables, for instance p and T . Experiments show that the dependence on the pressure is negligible for liquids and gases, a finding that agrees with the results of the kinetic theory of gases. As for the temperature dependence, the gas viscosity increases with temperature, because the faster random molecular motion enhances molecular momentum transport. For liquids, the intermolecular forces contribute fundamentally to the viscous stresses. With increasing temperatures the intermolecular distance also increases, causing the intermolecular force to drop and therefore producing a decrease in the viscosity (we have all seen how the oil viscosity drastically decreases as the cooking pan is heated on the stove). For both liquids and gases, so long as the temperature does not vary significantly, one may correctly assume that the viscosity coefficients remain constant, an assumption to be employed in the following development. As a final comment, it is worth mentioning that the bulk viscosity coefficient μ_B is identically zero for monoatomic gases, as indicated by the kinetic theory of gases.

Force and torque on a solid submerged in a fluid

The resultant force \bar{F} of the pressure and viscosity forces acting on a surface Σ of normal vector \bar{n} pointing towards the fluid can be computed from (3.22) and (3.28) to yield

$$\bar{F} = - \int_{\Sigma} p\bar{n}d\sigma + \int_{\Sigma} \bar{\tau}' \cdot \bar{n}d\sigma. \quad (3.32)$$

Similarly, the corresponding moment \bar{M} of the pressure and viscosity forces with respect to a given point \bar{x}_o is given by

$$\bar{M} = - \int_{\Sigma} (\bar{x} - \bar{x}_o) \wedge p\bar{n}d\sigma + \int_{\Sigma} (\bar{x} - \bar{x}_o) \wedge \bar{\tau}' \cdot \bar{n}d\sigma. \quad (3.33)$$

These equations can be used, in particular, to determine the force and torque that a fluid exerts on an solid body in contact with the fluid, with the surface Σ representing in that case the wetted surface of the body with the normal unit vector \bar{n} pointing towards the fluid.

The momentum equation

Newton's second law applied to a fluid volume states that the rate of change of the momentum equals the resultant force associated with the surface and volume forces acting on the fluid

$$\frac{d}{dt} \left[\int_{V_f(t)} \rho \bar{v} dV \right] = \int_{\Sigma_f(t)} \bar{\tau} \cdot \bar{n} d\sigma + \int_{V_f(t)} \rho \bar{f}_m dV. \quad (3.34)$$

Expressing the stress tensor $\bar{\tau}$ according to (3.28) and using Reynolds transport theorem (3.8) to extend the above equation for an arbitrary control volume $V_c(t)$ yields

$$\frac{d}{dt} \left[\int_{V_c(t)} \rho \bar{v} dV \right] + \int_{\Sigma_c(t)} \rho \bar{v} (\bar{v} - \bar{v}_c) \cdot \bar{n} d\sigma = - \int_{\Sigma_c(t)} p \bar{n} d\sigma + \int_{\Sigma_c(t)} \bar{\tau}' \cdot \bar{n} d\sigma + \int_{V_c(t)} \rho \bar{f}_m dV. \quad (3.35)$$

The above equation states that the rate of change of the momentum contained in a given control volume equals the sum of four contributions: the convective influx of momentum through its bounding surface $-\int_{\Sigma_c(t)} \rho \bar{v} (\bar{v} - \bar{v}_c) \cdot \bar{n} d\sigma$, the resultant pressure force $-\int_{\Sigma_c(t)} p \bar{n} d\sigma$, the resultant viscous force $\int_{\Sigma_c(t)} \bar{\tau}' \cdot \bar{n} d\sigma$ and the resultant mass force $\int_{V_c(t)} \rho \bar{f}_m dV$. Note that, for a fixed control volume, momentum conservation reduces to

$$\left[\int_{V_o} \frac{\partial \rho \bar{v}}{\partial t} dV \right] + \int_{\Sigma_o} \rho \bar{v} \bar{v} \cdot \bar{n} d\sigma = - \int_{\Sigma_o} p \bar{n} d\sigma + \int_{\Sigma_o} \bar{\tau}' \cdot \bar{n} d\sigma + \int_{V_o} \rho \bar{f}_m dV. \quad (3.36)$$

Angular momentum equation

Newton's second law can be alternatively expressed for the angular momentum in the following terms: the rate of variation of the angular momentum of a system equals the momentum of the external forces acting on it. For a fluid particle of mass ρdV located at \bar{x} the angular momentum with respect to a given position \bar{x}_o is given by $(\bar{x} - \bar{x}_o) \wedge \rho \bar{v} dV$. Therefore, when the overall fluid volume is considered, angular momentum conservation provides the integral equation

$$\frac{d}{dt} \left[\int_{V_f(t)} \rho (\bar{x} - \bar{x}_o) \wedge \bar{v} dV \right] = \int_{\Sigma_f(t)} (\bar{x} - \bar{x}_o) \wedge (\bar{\tau} \cdot \bar{n}) d\sigma + \int_{V_f(t)} \rho (\bar{x} - \bar{x}_o) \wedge \bar{f}_m dV, \quad (3.37)$$

which can be written for an arbitrary control volume $V_c(t)$ in the form

$$\begin{aligned} \frac{d}{dt} \left[\int_{V_c(t)} \rho (\bar{x} - \bar{x}_o) \wedge \bar{v} dV \right] + \int_{\Sigma_c(t)} \rho [(\bar{x} - \bar{x}_o) \wedge \bar{v}] [(\bar{v} - \bar{v}_c) \cdot \bar{n}] d\sigma = \\ - \int_{\Sigma_c(t)} (\bar{x} - \bar{x}_o) \wedge (p \bar{n}) d\sigma + \int_{\Sigma_c(t)} (\bar{x} - \bar{x}_o) \wedge (\bar{\tau}' \cdot \bar{n}) d\sigma + \int_{V_c(t)} \rho (\bar{x} - \bar{x}_o) \wedge \bar{f}_m dV. \end{aligned} \quad (3.38)$$

Energy conservation

Heat transfer by conduction

We have seen above that the existence of strain rates in the flow field results in the appearance of viscous stresses between adjacent portions of fluid. Likewise, the existence of temperature

gradients gives rise to a heat flux (conduction) whose microscopic origin lies - as that of the viscous stresses- in the random molecular motion, with the work of intermolecular forces also contributing non-negligibly in the case of liquids. As expected, the amount of energy transported is proportional to the surface considered, so that the heat flux (energy per unit time) through a surface element $d\sigma$ oriented perpendicular to the unit vector \bar{n} is given by

$$q_n(\bar{n}, \bar{x}, t)d\sigma, \quad (3.39)$$

where q_n represents the amount of energy transferred per unit surface per unit time in the direction \bar{n} . In the convention adopted, q_n is positive if the energy is transferred to the fluid element to which \bar{n} points from the fluid located on the side which \bar{n} points away from.

To investigate the dependence of q_n on the orientation let us consider once more the tetrahedral fluid element shown in Fig. 3.2. When its size becomes vanishingly small energy conservation can be shown to reduce *in the first approximation* to a balance between the heat fluxes through the four faces of the tetrahedron

$$q_n dA = q_1 dA_1 + q_2 dA_2 + q_3 dA_3. \quad (3.40)$$

In the notation adopted, $q_n dA$ is the flux through the oblique face with orientation \bar{n} and q_i denotes the heat flux per unit surface through a plane perpendicular to \bar{e}_i , so that $-q_i dA_i$ is the energy added per unit time to the tetrahedron due to heat conduction through its face perpendicular to \bar{e}_i . Using the geometrical relationship $dA_i = n_i dA$ and dividing the resulting equation by dA yields finally

$$q_n = \bar{q} \cdot \bar{n}, \quad (3.41)$$

showing that the heat flux in direction $\bar{n} = (n_1, n_2, n_3)$ can be expressed in terms of a vector $\bar{q} = (q_1, q_2, q_3)$, termed the heat-flux vector, whose three components are in general functions of the position and time considered.

Extending (3.39) to the computation of the heat flux through a surface Σ yields

$$\int_{\Sigma} \bar{q} \cdot \bar{n} d\sigma. \quad (3.42)$$

For a closed surface whose interior is free from localized heat sources Gauss formula (2.9) provides

$$\int_{\Sigma} \bar{q} \cdot \bar{n} d\sigma = \int_V \nabla \cdot \bar{q} dV, \quad (3.43)$$

where V is the volume enclosed by Σ . Consideration of the limit of very small volumes reveals in this case that $\nabla \cdot \bar{q}$ is the rate of conductive heat loss per unit volume (energy that abandons the unit volume per unit time due to heat conduction).

Fourier law

The heat-flux vector can be computed according to Fourier law

$$\bar{q} = -k \nabla T, \quad (3.44)$$

stating a linear proportionality between the heat-flux vector and the local temperature gradient, with the minus sign indicating that the energy is transported from regions of higher temperature

to regions of lower temperature. The proportionality constant k , called thermal conductivity, is a thermodynamic property that depends on the temperature but is independent of the pressure. In gases, where the conductive heat transfer is directly linked with the random motion of the molecules, a rise in temperature produces an increase in the thermal conductivity, so that, for instance, for air $k_a = 0.025 \text{ W/(m K)}$ at $T = 288 \text{ K}$ and $k_a = 0.03 \text{ W/(m K)}$ at $T = 368 \text{ K}$. Liquids behave differently, because the work of intermolecular forces play a nonnegligible role, causing the resulting thermal conductivity to exhibit in most liquids a slight decrease with increasing temperature. For water, however, the opposite response is observed, so that, for instance, $k_w = 0.59 \text{ W/(m K)}$ at $T = 288 \text{ K}$ and $k_w = 0.68 \text{ W/(m K)}$ at $T = 368 \text{ K}$.

The thermal diffusivity $\alpha = k/(\rho c_p)$ - or $\alpha = k/(\rho c)$ for a liquid- constitutes a useful measure of heat transfer by conduction. Its dimensions (squared length per unit time) are those of the kinematic viscosity ν defined above, so that the ratio $Pr = \nu/\alpha$ is a nondimensional number, called the Prandtl number, that serves to compare the molecular transport rates of momentum and heat for a given fluid. In fluids with $Pr \gg 1$, such as lubricating oils, the transport of momentum is much more effective than that of heat. Conversely, fluids with $Pr \ll 1$ may transport heat at a rate that is much more rapid than that of momentum transport, that being for instance the case of liquid metals like mercury, for which the presence of free electrons provides a very effective means for energy transfer. For gases, the Prandtl number is of order unity (e.g., $Pr = 0.72$ for air in a wide range of temperatures), while for water the value, dependent on the temperature, tends to be somewhat larger than unity ($Pr = 8.14$ at $T = 288 \text{ K}$ and $Pr = 1.82$ at $T = 368 \text{ K}$).

The energy equation

As discussed previously, the energy of a fluid (per unit mass) can be expressed as the sum of the kinetic energy associated with the fluid velocity, $|\bar{v}|^2/2$, and the internal energy e . The total energy contained in a fluid volume V_f is therefore given by $\int_{V_f} \rho(e + |\bar{v}|^2/2)dV$. According to the first law of thermodynamics, this quantity may change due to heat addition -by heat conduction- through the bounding fluid surface $\Sigma_f(t)$ or by chemical reaction or radiation inside the fluid volume. Additionally, it might also change due to the mechanical work exerted by mass or surface forces. This rate balance can be expressed through the integral equation

$$\begin{aligned} \frac{d}{dt} \left[\int_{V_f(t)} \rho(e + |\bar{v}|^2/2)dV \right] &= - \int_{\Sigma_f(t)} p\bar{v} \cdot \bar{n}d\sigma + \int_{\Sigma_f(t)} \bar{v} \cdot \bar{\tau}' \cdot \bar{n}d\sigma \\ &+ \int_{V_f(t)} \rho\bar{f}_m \cdot \bar{v}dV - \int_{\Sigma_f(t)} \bar{q} \cdot \bar{n}d\sigma + \int_{V_f(t)} (Q_c + Q_r) dV, \end{aligned} \quad (3.45)$$

where Q_c and Q_r represent, respectively, the rates of heat release (energy per unit volume per unit time) due to chemical reaction and radiation. Using Reynolds transport theorem enables

the above equation to be written for an arbitrary control volume in the form

$$\begin{aligned}
 \frac{d}{dt} \left[\int_{V_c(t)} \rho(e + |\bar{v}|^2/2) dV \right] + \int_{\Sigma_c(t)} \rho(e + |\bar{v}|^2/2) (\bar{v} - \bar{v}_c) \cdot \bar{n} d\sigma &= - \int_{\Sigma_c(t)} p \bar{v} \cdot \bar{n} d\sigma \\
 + \int_{\Sigma_c(t)} \bar{v} \cdot \bar{\tau}' \cdot \bar{n} d\sigma + \int_{V_c(t)} \rho \bar{f}_m \cdot \bar{v} dV - \int_{\Sigma_c(t)} \bar{q} \cdot \bar{n} d\sigma + \int_{V_c(t)} (Q_c + Q_r) dV. & \quad (3.46)
 \end{aligned}$$

Note that, in computing Q_c , one needs to consider separate conservation equations to compute the fluid composition, with account taken of chemical reactions occurring between the different chemical species. On the other hand, the computation of $Q_r = -\nabla \cdot \bar{q}_r$ from the radiative heat flux \bar{q}_r requires consideration of separate equations for radiation, accounting for local emission and absorption, with simplified limits arising in optically thin and optically thick environments. All these complexities are clearly beyond the scope of the present introductory course and are no longer considered here.