1 Introduction

Unlike rocket engines, airbreathing engines use the oxygen of the surrounding air for the combustion process, so that they do not need to carry the oxidant. Since the fuel-to-air mass ratio needed in stoichiometric combustion is typically very small, most of the mass flow through the engine is provided by the air stream, which is taken from the surrounding atmosphere through a diffuser. The main types of airbreathing engines, to be analyzed below, are shown schematically in Fig. 1.

Figure 1: Schematic view of the different types of airbreathing engines.

2 The thrust equation

A control-volume analysis can be used to derive an expression for the thrust generated by the engine. We consider first a turbojet engine attached to an airplane at takeoff, when the engine is surrounded by stagnant ambient air, the case represented on the left-hand side of Fig. 2. In steady operation mass conservation ($\int_{S_e} \rho v \cdot n \, dS = 0$) provides

$$\dot{m}_e = \dot{m}_a + \dot{m}_f,$$

indicating that the mass flow rate at the exit section is the sum of the air entrained by the engine and the fuel injected into the combustion chamber. The air flow is brought from all around the airplane. Since
the air velocity is inversely proportional to the square of the distance to the engine inlet, as follows from continuity, the momentum flux of the air entering the control volume is very small. If its contribution is neglected when writing the horizontal component \((x)\) of the momentum equation

\[
\int_{S_c} \rho v_x v \cdot n \, dS = -\int_{S_c} (p - p_a)n_x \, dS + T, \tag{2}
\]

it follows that

\[
T = \dot{m}_e u_e + (p_e - p_a)A_e = \dot{m}_a (1 + f) u_e + (p_e - p_a)A_e, \tag{3}
\]

a result analogous to that obtained previously for rocket engines. Equation (3) has been written in terms of the fuel-air ratio \(f = \dot{m}_f / \dot{m}_a\). As previously mentioned, this is a very small quantity, although its upper theoretical limit is \(f \approx 1/15\), corresponding to stoichiometric hydrocarbon-air combustion, typical values are significantly smaller, as needed to limit the peak temperature of the gas exiting the combustor.

![Schematic view of a single-exhaust airbreathing engine at take off (left) and in flight (right).](image)

The above result is modified when the airplane is flying at speed \(u\), the case represented on the right-hand side of Fig. 2. The control volume is bound laterally by a stream surface, where the pressure is assumed to be \(p_a\). Since the air entering the engine has a momentum flux relative to the airplane \(\dot{m}_a u\), the momentum balance yields in this case the so-called thrust equation

\[
T = \dot{m}_e u_e - \dot{m}_a u + (p_e - p_a)A_e = \dot{m}_a [(1 + f) u_e - u] + (p_e - p_a)A_e. \tag{4}
\]

As in the case of rockets, the contribution of the pressure term is typically small and can be correspondingly neglected for many purposes.

3 Performance parameters

3.1 Specific thrust and TSFC

The thrust is proportional to the size of the engine, which enters in (4) through the air mass flow rate. A relative measure of the thrust that enables the comparison between different engines, regardless of their size, is the so-called specific thrust

\[
\frac{T}{\dot{m}_a} = (1 + f) u_e - u + (p_e - p_a)A_e/\dot{m}_a \quad \left[ \frac{\text{kN} \cdot \text{s}}{\text{kg}} \right]. \tag{5}
\]
Neglecting the pressure term and using the approximation $f \ll 1$ yields the simplified result

$$\frac{T}{\dot{m}_a} \simeq u_e - u,$$  \hspace{1cm} (6)

which reveals the key role of the exhaust speed in jet-propulsion applications. Note that the result further reduces to $T/\dot{m}_a \simeq u_e$ for the specific thrust at takeoff, a result that can be directly compared with the specific impulse of rocket engines.

Fuel consumption is probably the most important factor in the selection of an airbreathing engine for civilian transport applications. A useful measure of this aspect of the engine performance is given by the thrust-specific fuel consumption

$$\text{TSFC} = \frac{\dot{m}_f}{T} = \frac{f}{T/\dot{m}_a} \left[ \frac{\text{kg}}{\text{kN} \cdot \text{s}} \right],$$  \hspace{1cm} (7)

defined as the mass of fuel needed to produce the unit of impulse.

### 3.2 Engine efficiencies

Both the specific thrust and the TSFC have dimensions. A dimensionless description of the performance of the engine can be made in terms of normalized efficiencies, expressing ratios of power output to power input. The objective of the jet engine is to generate thrust, with an associated power that for an airplane with flight speed $u$ is given by $Tu$. The energy is obtained by burning the fuel with a mass consumption rate $\dot{m}_f$. Correspondingly, the energy released per unit time is $\dot{m}_f Q_R$, with $Q_R$ denoting the energy released per unit mass of fuel consumed (remember that, for rockets, the heat of reaction $Q_R$ was referred to the unit mass of propellant instead). An ideal propulsion system should be able to convert all of the thermal energy into propulsion energy. In real systems, however, $Tu < \dot{m}_f Q_R$, as measured by the so-called overall efficiency

$$\eta_o = \frac{Tu}{\dot{m}_f Q_R} < 1.$$  \hspace{1cm} (8)

Using the approximation (6) leads to

$$\eta_o \simeq \frac{\dot{m}_a u(u_e - u)}{\dot{m}_f Q_R} = \frac{u_e^2}{fQ_R u_e} \left( 1 - \frac{u}{u_e} \right).$$  \hspace{1cm} (9)

This expression can be used to show that, for a fixed $u_e$, the maximum overall efficiency $\eta_o = u_e^2/(4fQ_R)$ is reached for flight conditions such that $u = u_e/2$.

Conceptually, the conversion of the thermal power into propulsion power can be thought to occur in two sequential steps. In the first step, the thermal energy is used to increase the kinetic energy of the gas stream flowing through the engine. The efficiency of this first step is measured by comparing the increment in the flux of kinetic energy $\dot{m}_e u_e^2/2 - \dot{m}_a u^2/2$ to the thermal power through the so-called thermal efficiency

$$\eta_{th} = \frac{\dot{m}_e u_e^2/2 - \dot{m}_a u^2/2}{\dot{m}_f Q_R} < 1.$$  \hspace{1cm} (10)

In the second step, the increase of kinetic energy is used to produce thrust, as measured by the so-called propulsion efficiency

$$\eta_p = \frac{Tu}{\dot{m}_e u_e^2/2 - \dot{m}_a u^2/2} < 1.$$  \hspace{1cm} (11)

Clearly,

$$\eta_o = \eta_{th} \eta_p.$$  \hspace{1cm} (12)
as follows from the above definitions. It is worth pointing out that in the approximation (6) with $\dot{m}_e = \dot{m}_a(1 + f) \approx \dot{m}_a$ the propulsion efficiency reduces to

$$
\eta_p \approx \frac{\dot{m}_a(u_e - u)}{\dot{m}_a(u_e - u)(u_e + u)} = \frac{2u/u_e}{1 + u/u_e}.
$$

(13)

It can be seen that the maximum propulsion efficiency $\eta_p = 1$ is reached as $u \to u_e$. Although the system would be very efficient in this limit, as measured by $\eta_p$, the associated specific thrust would be very small, as follows from (6), with the result that one would need to increase $\dot{m}_a$ (and, therefore, the size of the engine) to provide a nonnegligible amount of thrust.

### 3.3 Aircraft range

The importance of the overall efficiency $\eta_o$ becomes clear when considering the range of an aircraft (i.e. the distance that an aircraft can travel with a given amount of fuel $m_f$ at takeoff). At cruise conditions the forces acting on the aircraft are in equilibrium, so that the thrust $T$ equals the drag $D$ and the lift $L$ equals the weight $m_A g$, where $m_A$ denotes the instantaneous mass of the aircraft. This equilibrium balance can be expressed in the form

$$
T = D = \frac{L}{L/D} = \frac{m_A g}{L/D},
$$

(14)

which can be used to write the thrust power

$$
T u = \frac{m_A g}{L/D} u = \eta_o \dot{m}_f Q_R,
$$

(15)

the last expression following from the definition (8). On the other hand, the simple differential equations

$$
\frac{ds}{dt} = u
$$

(16)

and

$$
\frac{dm_A}{dt} = -\dot{m}_f,
$$

(17)

relating the aircraft speed $u$ with the distance travelled by the aircraft $s$ and the total mass of the aircraft $m_A$ with the fuel consumption rate $\dot{m}_f$, can be combined to give

$$
ds = -\frac{u}{\dot{m}_f} dm_A = -\eta_o (\frac{L}{D}) \frac{Q_R}{g} \frac{dm_A}{m_A},
$$

(18)

where the value of $u/\dot{m}_f$ is evaluated from the second equation in (15). If the dimensionless factor $\eta_o L/D$ is assumed to remain constant during flight, a reasonable approximation, then the above equation can be integrated with initial condition $m_A = m_o$ to give

$$
s = \eta_o (\frac{L}{D}) \frac{Q_R}{g} \ln \left( \frac{m_o}{m_A} \right)
$$

(19)

for the distance traveled as a function of the decreasing aircraft mass $m_A$. The range

$$
s_{\text{max}} = \eta_o (\frac{L}{D}) \frac{Q_R}{g} \ln \left( \frac{m_o}{m_o - m_f} \right)
$$

(20)

can be correspondingly computed as the distance traveled after consuming all of the available fuel. Equation (20) reveals, in particular, that the range is linearly proportional to the overall efficiency.
4 Energy balance in propulsion systems

The analysis of the performance of a given jet engine involves separate consideration of its different components, including the diffuser, compressor, combustor, turbine, and nozzle. In all cases, the integral energy balance provides valuable information regarding the associated changes of stagnation enthalpy. For the analysis, we begin by formulating the energy equation (i.e. the first law of thermodynamics) in the general integral form

\[
\frac{d}{dt} \left[ \int_{V_c} \rho \left( e + \frac{|v|^2}{2} \right) dV \right] + \int_{S_c} \rho \left( e + \frac{|v|^2}{2} \right) (v - v_c) \cdot n dS = - \int_{S_c} p v \cdot n dS + \int_{S_b} v \cdot \tau \cdot n dS + \int_{V_c} \dot{q}_R dV. \tag{21}
\]

corresponding to a control volume \( V_c \) bounded by the control surface \( S_c \) moving with velocity \( v_c \).

The above equation is applied to the generic component represented in Fig. 3, including possibly internal moving parts (i.e. the rotating blades of a compressor or turbine moving locally with velocity \( v_b \)) as well as an internal region where combustion takes place, \( \dot{q}_R \) representing the heat-release rate per unit volume. The control surface \( S_c \) includes a fixed outer boundary with \( v_c = 0 \), defined by the inlet section \( S_i \), the lateral surface \( S_l \) (where \( v = 0 \)), and the exit section \( S_e \), and an internal surface \( S_b \) moving with the rotating blades, so that \( v = v_c = v_b \) on \( S_b \). For a system in steady operation, the accumulation term is negligible in (21). Neglecting the work done by viscous forces on the fixed outer boundary, because either \( v = 0 \) (at \( S_l \)) or the local Reynolds number is large (at \( S_i \) and \( S_e \)), along with the effect of buoyancy forces and heat conduction reduces (21) to

\[
\int_{S_i + S_e} \rho \left( e + \frac{|v|^2}{2} \right) v \cdot n dS = - \int_{S_i + S_e} p v \cdot n dS - \int_{S_b} p v_b \cdot n dS + \int_{S_b} v_b \cdot \tau \cdot n dS + \int_{V_c} \dot{q}_R dV. \tag{22}
\]

The heat release rate is related to the fuel mass flow rate by

\[
\int_{V_c} \dot{q}_R dV = \eta_b \dot{m}_f Q_R, \tag{23}
\]

involving a burning efficiency \( \eta_b \) that accounts for incomplete combustion and heat losses. The two integrals \(- \int_{S_b} p v_b \cdot n dS + \int_{S_b} v_b \cdot \tau \cdot n dS\) account for the work done on the fluid per unit time by the rotating blades. This “shaft power”

\[
P_s = - \int_{S_b} p v_b \cdot n dS + \int_{S_b} v_b \cdot \tau \cdot n dS \tag{24}
\]
is positive for compressors and negative for turbines. Rewriting (22) in terms of the enthalpy $h = e + p/\rho$ gives

$$\int_{S_{i}+S_{e}} \rho \left( h + \frac{|v|^2}{2} \right) v \cdot n \, dS = P_{s} + \eta_{b} \dot{m}_{f} Q_{R},$$

(25)

to be used in the following analysis. For diffusers and nozzles $P_{s} = \dot{m}_{f} = 0$, so that the above equation reduces to the condition that the stagnation enthalpy $h_{0} = h + \frac{1}{2}|v|^2$ is conserved, i.e.

$$h_{0e} = h_{0i} \quad \text{(diffusers and nozzles).}$$

(26)

For compressors and turbines the solution yields

$$\dot{m}(h_{0e} - h_{0i}) = P_{c} \quad \text{(compressor)}$$

(27)

and

$$\dot{m}(h_{0i} - h_{0e}) = P_{t} \quad \text{(turbine)},$$

(28)

where $P_{c}$ and $P_{t}$ are the compressor power and turbine power, respectively. Finally, for combustors we find that $(\dot{m}_{a} + \dot{m}_{f})h_{0f} - \dot{m}_{a}h_{0a,i} - \dot{m}_{f}h_{0f,i} = \eta_{b} \dot{m}_{f} Q_{R}$, where $h_{0f,i}$ and $h_{0a,i}$ are the stagnation enthalpies of the fuel and the air in their feed streams. In most cases, the relative contribution of the term $\dot{m}_{f}h_{0f,i}$ is negligibly small and can be correspondingly neglected, thereby reducing the energy balance to

$$(\dot{m}_{a} + \dot{m}_{f})h_{0e} - \dot{m}_{a}h_{0a,i} = \eta_{b} \dot{m}_{f} Q_{R} \quad \text{(combustors).}$$

(29)

Equations (26)–(29) will be useful in the following development.

## 5 Ramjets

The ramjet is the simplest air-breathing engine. As indicated in Fig. 4, it comprises three main components, namely, (i) a diffuser, designed to compress the air by decelerating the incoming stream to low Mach numbers, (ii) a combustion chamber, where the oxygen of the air reacts with fuel that is continuously injected, and (iii) a convergent-divergent nozzle, designed to expand the hot stream, thereby converting the hot slow gas stream into a high velocity jet. Since there are no moving parts, the design and maintenance of ramjets are particularly simple. Besides, because of the absence of turbine blades between the combustor and the nozzle, cooling limitations are less stringent, thereby enabling higher combustion temperatures. The main drawback of ramjets is their inability to provide thrust at takeoff. They are mainly suitable for supersonic-flight applications at moderately large Mach numbers $M \sim 3$. Their performance is hindered both at low Mach numbers (i.e. subsonic flight), when the compression provided by the diffuser becomes insufficient, and at high Mach numbers (i.e. hypersonic flight), when the gas heating in the diffuser, associated with the air-stream deceleration from $M \gg 1$ to $M \ll 1$ (through oblique shocks), limits the temperature increase in the combustor. For that reason, the scramjet -a variant of the ramjet involving supersonic combustion- is currently being considered instead as a better suited option for hypersonic-flight applications.

### 5.1 Analysis of ramjet performance

In analyzing the performance of the ramjet we must consider separately the gas evolution across the diffuser ($a \to 2$), the combustion chamber ($2 \to 4$), and the nozzle ($4 \to 6$). The objective is to obtain the

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1 Note that the change in stagnation enthalpy across turbines and compressors, where the flow is nearly isentropic, is due to the intrinsic unsteady character of the flow, associated with the motion of the blades.
specific thrust $T/\dot{m}_a$, the thrust-specific fuel consumption TSFC, and the engine efficiencies $\eta_o$, $\eta_{th}$, and $\eta_p$ for given ambient air conditions (i.e. $T_a$, $p_a$, and $a_a$) and given flight speed $u$ and associated flight Mach number $M = u/a_a$. The fuel-to-air ratio $f$ will be taken as an unknown variable, to be calculated from the condition that the peak temperature, found at the entrance of the nozzle, cannot exceed a maximum value, determined by the materials and cooling requirements of the nozzle walls. For simplicity, we shall consider that the nozzle is designed to provide

$$p_e = p_a,$$

so that the thrust (4) will be computed using

$$T = \dot{m}_a[(1 + f)u_e - u].$$

### 5.1.1 Simplifications to the combustor flow

The low-Mach-number conditions prevailing in the combustion chamber are to be accounted in simplifying the description, i.e. since $M_2 \sim M_4 \ll 1$, it follows that

$$T_{0_2} = T_2 \left(1 + \frac{\gamma - 1}{2} M_2^2\right) \sim T_2 \quad \text{and} \quad T_{0_4} = T_4 \left(1 + \frac{\gamma - 1}{2} M_4^2\right) \sim T_4$$

and also that

$$p_{0_2} = p_2 \left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{\frac{\gamma}{\gamma - 1}} \sim p_2 \quad \text{and} \quad p_{0_4} = p_4 \left(1 + \frac{\gamma - 1}{2} M_4^2\right)^{\frac{\gamma}{\gamma - 1}} \sim p_4.$$  

Furthermore, an order-of-magnitude analysis of the momentum balance equation indicates that, for high-Reynolds number flow, the relative spatial pressure differences scale with the square of the existing Mach number. Since $M_2 \sim M_4 \ll 1$ in the combustion chamber, it follows that $(p_2 - p_4)/p_2 \sim (p_{0_2} - p_{0_4})/p_{0_2} \ll 1$. We shall see below that these simplifications are no longer valid across the supersonic-combustion chamber of scramjets, where the pressure increases by a relative amount of order unity $(p_4 - p_2)/p_2 \sim 1$ as a result of the heat-release process.

### 5.1.2 Energy balance

The energetics of the diffuser can be evaluated by straightforward application of (26) to give

$$h_{0_2} = h_{0_a} = h_a + \frac{u^2}{2} \quad \rightarrow \quad \frac{h_{0_2}}{h_a} = \frac{T_{0_2}}{T_a} = 1 + \frac{\gamma - 1}{2} M^2.$$
Similarly, we find
\[ h_{0a} = \frac{u_e^2}{2} + h_e + \frac{u_e^2}{2} \rightarrow \frac{h_{0a}}{h_a} = \frac{T_{0a}}{T_a} = \frac{T_e}{T_a} (1 + \frac{\gamma - 1}{2} M_e^2) \] (35)
across the nozzle. Equation (29) can be used to write
\[ (\dot{m}_a + \dot{m}_f)h_{a4} - \dot{m}_a h_{0a} = \eta_b \dot{m}_f Q_R \rightarrow \frac{(1 + f)T_{0a}}{T_a} - \frac{T_{0a}}{T_a} = \frac{\eta_b f Q_R}{c_p T_a}, \] (36)
with \( T_{0a}/T_a = 1 + \frac{\gamma - 1}{2} M^2 \), as follows from (34). Since the peak temperature at the end of the combustion process \( T_4 \approx T_{0a} \) is taken as a known limiting parameter, as previously mentioned, we can solve the above equation to determine the fuel-to-air ratio
\[ f = \frac{(T_{0a})}{\eta_b Q_R (c_p T_a)} - \frac{(T_{0a})}{T_a} \] (37)
in terms of known quantities (i.e., the peak-to-ambient temperature ratio \( T_{0a}/T_a \), the flight Mach number \( M \), the burning efficiency \( \eta_b \), and the ratio \( Q_R/(c_p T_a) \) of the heat of reaction to the ambient enthalpy).

5.1.3 Performance parameters

It is possible to relate all of the different ramjet performance parameters to the (still unknown) value of the exit Mach number \( M_e \). To that end, we begin by using (35) to write
\[ u_e = M_e a_e = a_0 M_e \sqrt{\frac{T_e}{T_a}} = a_0 \frac{(T_{0a}/T_a)^{1/2} M_e}{(1 + \frac{\gamma - 1}{2} M_e^2)^{1/2}}, \] (38)
which can be used in (31) to provide
\[ \frac{T}{\dot{m}_a} = a_0 \left[ (1 + f) \frac{(T_{0a}/T_a)^{1/2} M_e}{(1 + \frac{\gamma - 1}{2} M_e^2)^{1/2}} - M \right] \] (39)
for the ramjet specific thrust, with the corresponding
\[ \text{TSFC} = \frac{f}{\frac{T}{\dot{m}_a}} \] (40)
evaluated from (7) once the fuel-to-air ratio \( f \) is computed from (37). Similarly, one can derive from the definitions (8), (10), and (11) the expressions
\[ \eta_o = \frac{(\gamma - 1)M}{f Q_R (c_p T_a)} \left[ (1 + f) \frac{(T_{0a}/T_a)^{1/2} M_e}{(1 + \frac{\gamma - 1}{2} M_e^2)^{1/2}} - M \right] \] (41)
\[ \eta_{th} = \frac{(\gamma - 1)/2}{f Q_R (c_p T_a)} \left[ (1 + f) \frac{(T_{0a}/T_a) M_e^2}{(1 + \frac{\gamma - 1}{2} M_e^2)^{1/2}} - M^2 \right] \] (42)
\[ \eta_p = 2M \left[ (1 + f) \frac{(T_{0a}/T_a)^{1/2} M_e}{(1 + \frac{\gamma - 1}{2} M_e^2)^{1/2}} - M \right] \right] \left[ (1 + f) \frac{(T_{0a}/T_a) M_e^2}{(1 + \frac{\gamma - 1}{2} M_e^2)} - M^2 \right] \] (43)
for the engine efficiencies. In order to evaluate (38)–(43) we need to find the exit Mach number $M_e$, following the procedure indicated below.

### 5.2 Ideal ramjets

It is instructive to consider first the case of ideal ramjets, where we assume that the flow is isentropic in the diffuser and nozzle, so that $s_a = s_2$ and $s_4 = s_e$, and neglect pressure losses across the combustor, so that

$$p_2 = p_0 = p_4 = p_0_4.$$  \hfill (44)

The corresponding gas evolution is represented in the $h - s$ diagram shown in Fig. 5, which for illustrative purposes includes the approximation $h_0_4 - h_0_2 = h_4 - h_2 \simeq \eta f Q_R$, stemming from (65) when $f \ll 1$.

![Figure 5: The enthalpy-entropy variation for the gas flow across the ideal ramjet.](image)

Since the flow is both steady and isentropic in the diffuser and nozzle, it follows that $p_0_4 = p_0_2$ and that $p_0 = p_0_e$, which can be combined with (44) and the condition $p_e = p_a$ to give

$$p_0_a = p_0_e \rightarrow p_a \left(1 + \frac{\gamma - 1}{2} M^2\right)^{(\gamma/(\gamma - 1))} = p_a \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\gamma/(\gamma - 1)},$$ \hfill (45)

leading to the conclusion that $M_e = M$!! This is of course a direct consequence of the conservation of stagnation pressure across the ramjet: a deceleration/compression followed by an acceleration/expansion to the same initial pressure leads to a stream with the same Mach number. As inferred from (31) with $f \ll 1$, the generation of thrust requires that $u_e = M_e a_e > u = M a_a$. For the ideal ramjet $M_e = M$ and $T_e = T_0_4/(1 + \frac{\gamma - 1}{2} M^2)$, the latter following from (35), so that positive thrust is generated provided that $M < (\frac{2}{\gamma - 1})^{1/2}(T_0_4/T_a - 1)^{1/2}$, thereby placing an upper limit to the range of flight Mach numbers where ramjets can operate successfully.

Using the result $M_e = M$ in (39)–(43) allows us to evaluate the performance of the ideal ramjet. The variation with flight Mach number of $T/m_a$, TSFC, $\eta_o$, $\eta_p$, and $\eta_h$ are shown as solid curves in Fig. 6 for an ideal Ramjet with $\gamma = 1.3$, $T_0_4/T_a = 12$, $Q_R/(c_p T_a) = 200$, $\eta_h = 1$, and $a_a = 300$ m/s. The results indicate that the specific thrust, measuring the thrust per unit size of engine, reaches a maximum at $M \simeq 2.9$ and decreases for higher values, eventually vanishing at $M \simeq 8.5$. By way of contrast, the overall efficiency, which determines the range, increases continuously for increasing values of $M$. Also of
interest is that the value of TSFC remains almost constant over a wide range of intermediate values of $M$.

Figure 6: Variation with the flight Mach number of the performance parameters of a ramjet. Results are evaluated with $\gamma = 1.3$, $T_0 / T_a = 12$, $Q_R / (c_p T_a) = 200$, $\eta_b = 1$, and $a_a = 300$ m/s for an ideal ramjet with no aerodynamic losses (solid curves) and for a real ramjet with adiabatic efficiencies $\eta_d = 0.8$ and $\eta_n = 0.98$ and combustion pressure ratio $r_c = 0.9$ (dashed curves).

5.3 Effects of aerodynamic losses

The above analysis, assuming isentropic flow across the diffuser and nozzle and uniform pressure across the combustor, provides an idealized picture of the ramjet performance. In reality, however, there are nonnegligible losses of stagnation pressure (and associated increases of entropy) across the diffuser, where the deceleration of the air stream takes place through a series of oblique shocks, and also across the nozzle, related mostly to the action of the viscous forces in the wall boundary layer. Also, the flow across the combustor involves a small but finite pressure loss, needed to overcome the drag forces on flame holders and fuel injectors and the viscous forces on the walls, so that $p_4 < p_2$. Because of these cumulative aerodynamic losses the exit velocity $u_e$ is smaller than that predicted by the ideal-ramjet model. This can be seen clearly in Fig. 7, which compares the $h$ – $s$ evolution of the gas in the ideal ramjet of Fig. 5 with that of a real ramjet with $s_2 > s_a$, $p_4 < p_2$, and $s_e > s_4$.

The simplest way to quantify these aerodynamic losses is by defining the pressure ratios across the diffuser, combustor, and nozzle

$$r_d = \frac{p_{02}}{p_{04}} < 1, \quad r_c = \frac{p_4}{p_2} = \frac{p_{04}}{p_{02}} < 1, \quad r_n = \frac{p_{0b}}{p_{04}} = \frac{p_{0e}}{p_{04}} < 1.$$ (46)

Since

$$\frac{p_{0b}}{p_{0a}} = r_d r_c r_n = \left( \frac{1 + \frac{\gamma - 1}{2} M_e^2}{1 + \frac{\gamma - 1}{2} M^2} \right)^{\frac{\gamma}{\gamma - 1}} < 1$$ (47)

the resulting Mach number at the exit

$$M_e = \left( \frac{2}{\gamma - 1} \right)^{1/2} \left[ (r_d r_c r_n)^{\frac{\gamma - 1}{\gamma}} \left( 1 + \frac{\gamma - 1}{2} M^2 \right) - 1 \right]^{1/2}.$$ (48)
For diffusers and nozzles, and also for compressors and turbines, to be discussed below, the departures from isentropic behavior are usually measured in terms of the so-called adiabatic efficiencies. For a diffuser, the adiabatic efficiency

\[ \eta_d = \frac{h_{2s} - h_a}{h_2 - h_a}. \]  

is defined as the ratio of the enthalpy change involved in an isentropic compression from \( p_a \) to \( p_2 \) to the actual change occurring in the real system, with \( 2s \) denoting the downstream isentropic state, as indicated on the left-hand side of Fig. 8. Since \( h_2 - h_a = \frac{u^2}{2} \), dividing by \( h_a \) numerator and denominator of (49) and using the condition that the evolution \( a \to 2s \) is isentropic with \( p_2 = p_{02} \) provides

\[ \eta_d = \left( \frac{p_{02}}{p_a} \right)^{\frac{\gamma - 1}{2}} - 1 \rightarrow \frac{p_{02}}{p_a} = \frac{p_2}{p_a} = \left( 1 + \eta_d \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma - 1}{\gamma - 1}}. \]  

Figure 7: Comparison between the enthalpy-entropy variation for the gas flow across an ideal ramjet (blue) with that of a real ramjet with \( s_2 > s_a, p_4 < p_2 \), and \( s_e > s_4 \) (red).

is smaller than \( M \).

Figure 8: Comparison between the ideal and real gas evolution across diffusers and nozzles used in the definitions of their corresponding adiabatic efficiencies.
Similarly, for a nozzle the adiabatic efficiency
\[
\eta_n = \frac{h_4 - h_6}{h_4 - h_{6s}}.
\]
is defined as the ratio of the enthalpy change \(h_4 - h_6\) involved in the expansion from \(p_4 = p_{04}\) to \(p_6 = p_a\) to the change \(h_4 - h_{6s}\) that would be involved should the process be isentropic, with \(6s\) denoting the downstream isentropic state, as indicated on the right-hand side of Fig. 8. Using \(h_4 - h_6 = u_e^2/2\) together with the conditions \(T_4 = T_{04}, (h_{6s}/h_4) = (T_{6s}/T_4) = (p_a/p_4)^{(\gamma - 1)/\gamma}\), and \(T_{04}/T_6 = T_{06}/T_6 = 1 + \frac{\gamma - 1}{2} M_e^2\) leads to
\[
\eta_n = \frac{u_e^2/2}{c_p T_{04}(1 - h_{6s}/h_4)} = \frac{\frac{\gamma - 1}{2} M_e^2}{(1 + \frac{\gamma - 1}{2} M_e^2) \left[ 1 - \left( \frac{p_a}{p_4} \right)^{(\gamma - 1)/\gamma} \right]}.
\]

Using \(p_4/p_a = r_c (p_2/p_a)\) with \(p_2/p_a\) evaluated from (50) as a function of \(M\) and solving the above equation for \(M_e\) yields
\[
M_e = \left( \frac{2}{\gamma - 1} \right)^{1/2} \left[ \frac{r_c^{(\gamma - 1)/\gamma} \left( 1 + \eta_d \frac{\gamma - 1}{2} M_e^2 \right) - 1}{\left( 1 - \eta_d r_c^{(\gamma - 1)/\gamma} \right) \left( 1 + \eta_d \frac{\gamma - 1}{2} M_e^2 \right) + 1} \right]^{1/2},
\]
which reduces to \(M_e = M\) when \(\eta_d = r_c = \eta_n = 1\). Equation (53) is used, together with (39)–(43), to generate the dashed curves of Fig. 6, representing the degraded performance of a real ramjet with \(\eta_d = 0.8, \eta_n = 0.98,\) and \(r_c = 0.9\). As can be seen, although there exist quantitative differences between the real and the ideal ramjet, the ideal model qualitatively captures the trends of the different performance parameters, with the largest departures observed in the predictions of TSFC.

5.4 High-speed combustion

The previous analysis showed that the specific thrust reaches a maximum for a flight mach number \(M \simeq 2.9\) and decays for larger values of \(M\). The reason for this decay is that for \(M \gg 1\) the compression across the diffuser places the system at temperatures \(T_{02}\) close to the maximum value allowed for \(T_{04}\), thereby limiting the enthalpy gain \(h_{04} - h_{02} \simeq h_4 - h_2\) across the combuster and, therefore, the resulting exit velocity \(u_e\). To overcome this limitation, one alternative for large \(M\) is to limit the deceleration/compression of the air stream in the diffuser, so that the final Mach number \(M_2\) remains supersonic, as done in supersonic combustion ramjets (scramjets). The corresponding analysis of their performance parallels that presented above. The main differences are that the approximations (32) and (33) do not apply and that the supersonic flow in the combuster involves large relative pressure differences of order unity that need to be accounted for, as indicated below.

The analysis of the combustion process when the Mach number is of order unity simplifies for a combustion chamber of constant section. As done earlier in section 4, we will use the subscripts \(i\) and \(e\) to denote the inlet and exit properties, which are related by the balances of mass, momentum, and energy. In particular, using the approximation \(f \ll 1\) when writing the energy equation (29)
\[
(1 + f) c_p T_{0e} - c_p T_{0i} = \eta_b f Q_R
\]
provides the expression
\[
\frac{T_{0e}}{T_{0i}} = 1 + \frac{\eta_b f Q_R}{c_p T_{0i}}
\]
for the jump in stagnation temperature. Neglecting friction along with the negligible momentum flux of the fuel stream reduces the integral momentum balance to \(p_i + \rho_i u_i^2 = p_e + \rho_e u_e^2\), which can be written in
the form

\[ \frac{p_e}{p_i} = 1 + \gamma M_i^2 \]  

(56)

relating the pressure jump with the inlet and exit Mach numbers (note, in particular, that if \(M_i \ll 1\) and \(M_e \ll 1\), then \(p_i \simeq p_e\), that being the case of ramjet combustors). If the combustor section is constant, then neglecting the mass of the fuel compared with the mass of air when writing the continuity equation \(\dot{m}_e = \dot{m}_a(1 + f) \simeq \dot{m}_a\) provides

\[ \rho_i u_i = \rho_e u_e \quad \rightarrow \quad \frac{T_e}{T_i} = \left( \frac{p_e}{p_i} \frac{M_e}{M_i} \right)^2. \]  

(57)

Using (56) in this last expression along with the definition \(T_0/T = 1 + \frac{\gamma - 1}{2} M^2\) leads to

\[ \frac{T_{0e}}{T_{0i}} = \left[ \frac{M_e(1 + \frac{\gamma - 1}{2} M_e^2)^{1/2}/(1 + \gamma M_e^2)}{M_i(1 + \frac{\gamma - 1}{2} M_i^2)^{1/2}/(1 + \gamma M_i^2)} \right]^2 = \frac{F^2(M_e)}{F^2(M_i)}, \]  

(58)

where the function

\[ F = \frac{M(1 + \frac{\gamma - 1}{2} M^2)^{1/2}}{1 + \gamma M^2} \]  

(59)

is represented in Fig. 9.

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
0.5 \\
0.45 \\
0.4 \\
0.35 \\
0.3 \\
0.25 \\
0.2 \\
0.15 \\
0.1 \\
0.05 \\
0 \\
0.05 \\
0.1 \\
0.15 \\
0.2 \\
0.25 \\
0.3 \\
0.35 \\
0.4 \\
0.45 \\
0.5 \\
\end{array}
\end{array}
\end{array}
\]

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array}
\end{array}
\end{array}
\]

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
F(M) \\
\gamma = 14 \\
\gamma = 12 \\
\end{array}
\end{array}
\end{array}
\]

Figure 9: The function \(F\) defined in (59).

Equations (55) and (58) can be used together with Fig. 9 to illustrate some of the characteristics of the flow. Since \(T_{0e} > T_{0i}\) as a result of the heat-release process, as indicated by (55), then according to (58) \(F(M_e)\) must be larger than \(F(M_i)\). Two cases present themselves:
• If the flow is initially subsonic ($M_i < 1$), then Fig. 9 reveals that the increase of $F$ is associated with a larger value of $M_e > M_i$ and, according to (56), a smaller value of $p_e < p_i$, i.e., subsonic combustion leads to an acceleration/expansion of the flow.

• Conversely, if the flow is initially supersonic ($M_i > 1$), then the increase of $F$ is associated with a smaller value of $M_e < M_i$ and a larger value of $p_e > p_i$, i.e. supersonic combustion leads to a deceleration/compression of the flow.

In both cases, the function $F$ is limited to a maximum value (e.g. $F \simeq 0.456$ for $\gamma = 1.4$), associated with the maximum increase in stagnation temperature that can be achieved before the flow chokes.

6 Turbojet engines

To enable takeoff capabilities and extend the range of flight conditions, the aerodynamic compressor (i.e. the diffuser) of the ramjet can be supplemented with a mechanical compressor. The power needed to move the compressor can be provided by a co-axial turbine, to be placed immediately downstream from the combustor. The resulting arrangement is the basis of the so-called turbojet, schematically represented in Fig. 10, with the corresponding enthalpy-entropy diagram shown in Fig. 11.

Figure 10: Schematic view of a turbojet with indication of the numbering used in the analysis of the flow.

6.1 Performance analysis

In analyzing the performance of a turbojet separate consideration must be given to the gas evolution across the diffuser ($a \rightarrow 2$), compressor ($2 \rightarrow 3$), combustor ($3 \rightarrow 4$), turbine ($4 \rightarrow 5$), and nozzle ($5 \rightarrow 6$). The expansion in the nozzle is assumed to be such that $p_6 = p_e = p_a$, so that the thrust simplifies to the expression given in (31). In typical systems the Mach number in the engine between the diffuser and the nozzle is very small, with the result that $T_{02}/T_2 = T_{03}/T_3 = T_{04}/T_4 = T_{05}/T_5 \simeq 1$ and $p_{02}/p_2 = p_{03}/p_3 = p_{04}/p_4 = p_{05}/p_5 \simeq 1$. The analysis assumes known ambient air conditions (i.e. $T_a$, $p_a$, and $a_a$), gas properties ($\gamma$ and $c_p$), combustion parameters ($\eta_b$ and $Q_R$), flight speed $u$ and associated flight Mach number $M = u/a_a$, compressor pressure ratio $P_{rc} = p_{03}/p_{02}$, and maximum allowable temperature $T_{04}$. The values of the specific thrust $T/\dot{m}_a$, thrust-specific fuel consumption $TSFC$, and engine efficiencies $\eta_o$, $\eta_{th}$, and $\eta_p$, depend on the unknown values of the fuel-to-air ratio $f$ and exit speed $u_e$, to be computed as indicated below. Formulas will be derived for a real turbojet with given values of the adiabatic efficiencies of the diffuser, compressor, turbine, and nozzle ($\eta_d$, $\eta_c$, $\eta_t$, $\eta_n$) and a given pressure ratio across the combustor $r_c = p_{04}/p_3 = p_{04}/p_{01}$, with formulas for the ideal turbojet obtained by substituting ($\eta_d = \eta_c = \eta_t = \eta_n = r_c = 1$).
Figure 11: The enthalpy-entropy variation for the gas flow across an ideal turbojet (left) and across a real turbojet (right).

### 6.1.1 Flow across the diffuser ($a \to 2$)

The analysis of this component is identical to that presented earlier in relation to the ramjet. Thus energy conservation leads to $h_0^a = h_0^2 \approx h_2$, which can be used to write

$$h_2 - h_a = u_2^2 / 2$$

and

$$\frac{T_0^a}{T_a} = \frac{T_2}{T_a} = 1 + \frac{\gamma - 1}{2} M^2,$$

while the pressure ratio reduces to

$$\frac{p_0^a}{p_a} = \frac{p_2}{p_a} = \left(1 + \eta_d \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$

involving the diffuser adiabatic efficiency $\eta_d$.

### 6.1.2 Flow across the compressor ($2 \to 3$)

The energy balance across the compressor (27) can be used to write

$$m_a (h_3^a - h_0^2) = P_c \quad \Rightarrow \quad h_0^3 - h_0^2 = h_3 - h_2 = P_c / m_a,$$

where $P_c$ is the compressor power. Departures from isentropic behavior are measured through the compressor adiabatic efficiency

$$\eta_c = \frac{h_{3s} - h_2}{h_3 - h_2}$$

defined as the enthalpy change involved in the isentropic compression from $p_2$ to $p_3$ divided by the actual enthalpy change (see the $h - s$ evolution shown on the left-hand side of Fig. 12). Since $h_{3s} / h_2 = (p_3 / p_2)^{\gamma - 1} = P_{r_c}^{\frac{\gamma - 1}{\gamma}}$, where $P_{r_c}$ is the known compressor pressure ratio, it follows that

$$\eta_c = \frac{h_{3s} - h_2}{h_3 - h_2} = \frac{P_{r_c}^{\frac{\gamma - 1}{\gamma}} - 1}{T_3 / T_2 - 1} \quad \Rightarrow \quad \frac{T_0^3}{T_0^2} = \frac{T_3}{T_2} = 1 + \frac{P_{r_c}^{\frac{\gamma - 1}{\gamma}} - 1}{\eta_c},$$

which naturally reduces to $T_3 / T_2 = (p_3 / p_2)^{\gamma - 1} = P_{r_c}^{\frac{\gamma - 1}{\gamma}}$ for the ideal compressor (i.e. $\eta_c = 1$).
6.1.3 Flow across the combustor (3 → 4)

The analysis parallels that presented earlier for the ramjet. Energy conservation provides

\[(\dot{m}_a + \dot{m}_f)h_{04} - \dot{m}_a h_{03} = \eta_b \dot{m}_f Q_R \quad \rightarrow \quad (1 + f) \frac{T_{04}}{T_a} - \frac{T_{03}}{T_a} = \eta_b f \frac{Q_R}{c_p T_a},\]

which can be combined with (60) and (64) to give

\[f = \frac{\left( \frac{T_{04}}{T_a} \right) - \left( 1 + \frac{\gamma - 1}{\eta_c} \right) \left( 1 + \frac{\gamma - 1}{2} M^2 \right)}{\eta_b \left( \frac{Q_R}{c_p T_a} \right) - \left( \frac{T_{04}}{T_a} \right)} \quad (66)\]

for the fuel-air ratio.

6.1.4 Flow across the turbine (4 → 5)

The energy balance across the turbine (28) gives

\[(\dot{m}_a + \dot{m}_f)(h_{04} - h_{05}) = P_t \quad (67)\]

relating the enthalpy decrease across the turbine \(h_{04} - h_{05} = h_4 - h_5\) with the turbine power. The pressure jump across the ideal turbine is given by the isentropic relation \((p_{05}/p_{04})^{\frac{\gamma - 1}{\gamma}} = T_{05}/T_{04}\). Departures from this isentropic behavior are characterized by the adiabatic efficiency

\[\eta_t = \frac{h_4 - h_5}{h_4 - h_{5s}} = \frac{1 - T_{05}/T_{04}}{1 - (p_{05}/p_{04})^{\frac{\gamma - 1}{\gamma}}} \quad \rightarrow \quad \left( \frac{p_{05}}{p_{04}} \right)^{\frac{\gamma - 1}{\gamma}} = 1 - \frac{1 - T_{05}/T_{04}}{\eta_t} \quad (68)\]

associated with the expansion from \(p_4 = p_{04}\) to \(p_5 = p_{05}\), as indicated on the right-hand side of Fig. 12.

6.1.5 Flow across the nozzle (5 → 6 = e)

Conservation of stagnation enthalpy \((h_{0_e} = h_{0_5})\) with \(h_{0_e} = h_e + \frac{u_e^2}{2}\) and \(h_{0_5} = h_5\) gives

\[h_5 - h_e = \frac{u_e^2}{2}. \quad (69)\]
On the other hand, from the definition of adiabatic efficiency,

\[ \eta_a = \frac{h_5 - h_6}{h_5 - h_{0_s}} = \frac{u_e^2/2}{c_p T_0 \left[ 1 - (p_e/p_5)^{\gamma-1} \right]}, \]  
(70)

with \( p_e = p_a, \) \( p_5 = p_{0_s}, \) and \( c_p = \gamma R_g/(\gamma - 1) \) it follows that

\[ \frac{u_e^2}{2} = \frac{\eta_a a^2}{\gamma - 1} \left( \frac{T_{0_5}}{T_{0_4}} \right) \left( \frac{T_{0_4}}{T_a} \right) \left[ 1 - \frac{1}{\left(1 - \frac{T_{0_5}/T_{0_4}}{\eta_c} \right) r_c^{\gamma - 1} P_r^{\gamma} \left(1 + \eta_c \frac{\gamma - 1}{2} M^2\right)} \right], \]  
(71)

where \( a^2 = \gamma R_g T_a. \) Substituting in the above equation (61), (68) and the known pressure jumps \( P_r = p_{0_3}/p_{0_2} \) and \( r_c = p_{0_4}/p_{0_3} \) gives

\[ \frac{u_e^2}{2} = \frac{\eta_a a^2}{\gamma - 1} \left( \frac{T_{0_5}}{T_{0_4}} \right) \left( \frac{T_{0_4}}{T_a} \right) \left[ 1 - \frac{1}{\left(1 - \frac{T_{0_5}/T_{0_4}}{\eta_c} \right) r_c^{\gamma - 1} P_r^{\gamma} \left(1 + \eta_c \frac{\gamma - 1}{2} M^2\right)} \right], \]  
(72)

which determines the gas speed at the exit \( u_e \) in terms of the (still) unknown temperature jump \( T_{0_5}/T_{0_4}. \)

6.1.6 The temperature jump across the turbine

Since the purpose of the turbine is to move the compressor, it follows that

\[ P_t = P_c. \]  
(73)

Substituting (62) and (67) into the above equation provides

\[ (\dot{m}_a + \dot{m}_f)(h_{0_4} - h_{0_5}) = \dot{m}_a (h_{0_4} - h_{0_2}) \rightarrow \]  

\[ (1 + f) (T_{0_4} - T_{0_5}) = T_{0_1} - T_{0_2}, \]  
(74)

which can be rearranged with use made of (60) and (64) to give

\[ \frac{T_{0_5}}{T_{0_4}} = 1 - \left( \frac{T_{0_5}}{T_{0_4}} - 1 \right) \frac{T_{0_4}}{T_a} = 1 - \left( \frac{P_r^{\gamma - 1} - 1}{\eta_c (1 + f)} \right) \frac{T_{0_4}}{T_a}, \]  
(75)

for the temperature jump across the turbine. For many purposes, the above equation can be written in the simplified form

\[ \frac{T_{0_5}}{T_{0_4}} = 1 - \left( \frac{P_r^{\gamma - 1} - 1}{\eta_c T_{0_4}/T_a} \right) \frac{T_{0_4}}{T_a}, \]  
(76)

by using the approximation \( f \ll 1 \) (the mass flow rate across the turbine is assumed to be equal to the mass flow rate across the compressor). Equation (75) (or (76)) can be used in (72) to compute \( u_e. \) For the ideal turbojet with \( f \ll 1, \) the solution can be seen to simplify to

\[ \frac{u_e^2}{2} = \frac{a^2}{\gamma - 1} \left( \frac{T_{0_4}}{T_a} \right) \left[ 1 - \frac{(P_r^{\gamma - 1} - 1)(1 + \frac{\gamma - 1}{2} M^2)}{T_{0_4}/T_a} \right], \]  
(77)

which shows explicitly the dependence on the peak temperature \( T_{0_4}, \) compressor pressure ratio \( P_r, \) and flight Mach number \( M. \) The reader can verify that for \( P_r = 1 \) this last equation reduces to \( M_e = M, \) as it should, since in the absence of mechanical compression the ideal turbojet reduces to an ideal ramjet.
6.2 Turbojet performance

Equations (66) and (72) (supplemented with (75) or (76) for the computation of $T_{05}/T_{04}$) can be used to evaluate the fuel-to-air ratio $f$ and the exit gas speed $u_e$, to be employed in evaluating the performance of the turbojet through

$$\frac{T}{\dot{m}_a} = (1 + f)u_e - u, \quad \text{TSFC} = \frac{f}{(T/\dot{m}_a)}$$

and

$$\eta_o = \frac{(T/\dot{m}_a)u}{fQ_R}, \quad \eta_{th} = \frac{(1 + f)u_e^2/2 - u^2/2}{fQ_R}, \quad \eta_p = \frac{(T/\dot{m}_a)u}{(1 + f)u_e^2/2 - u^2/2}.$$  

The typical variation of the different performance parameters with the compressor pressure ratio is shown in Fig. 13 for a turbojet in supersonic flight at $M = 1.5$. Results are given for an ideal turbojet (solid curves) and for a real turbojet (dashed curves). Note that the limiting values for $P_{rc} = 1$ correspond to those of a ramjet at the same flight conditions.

Figure 13: Variation with the compressor pressure ratio $P_{rc}$ of the performance parameters of a turbojet flying at $M = 1.5$. Results are evaluated with $\gamma = 1.3$, $T_{04}/T_a = 10$, $Q_R/(c_pT_a) = 200$, $\eta_b = 1$, and $a_a = 300$ m/s for an ideal turbojet with no aerodynamic losses (solid curves) and for a real turbojet with adiabatic efficiencies $\eta_d = 0.9$, $\eta_c = 0.85$, $\eta_h = 0.85$ and $\eta_m = 0.95$ and combustion pressure ratio $r_c = 0.97$ (dashed curves).

The curves for specific thrust in Fig. 13 reveal the existence of a maximum at an intermediate value of $P_{rc}$. To further explore this aspect of the problem, additional results are shown in Fig. 14 for an ideal turbojet. The left plot shows the important effect of the peak temperature on the turbojet performance. The right-hand side plot shows that the performance degrades with increasing flight Mach number $M$. It is of interest that the optimum $P_{rc}$ at which the specific thrust reaches its maximum value decreases with increasing $M$. For $M = 4$ the maximum thrust is found at $P_{rc} = 1$, indicating that for these high speeds a ramjet is more efficient than a turbojet.

7 Turbofan engines

The performance of the turbojet can be improved in a modified design where part of the air bypasses the combustor, that being the idea underlying turbofan engines. The typical system is shown in Fig. 15. The total air mass flow rate ingested by the engine is $(1 + B)\dot{m}_a$, with $\dot{m}_a$ representing the air flow rate through
Figure 14: Variation of the specific thrust with the compressor pressure ratio $P_{rc}$ for a turbojet flying at $M = 1.5$ with different peak temperatures (left plot) and for a turbojet with $T_0/T_a = 10$ flying at different Mach numbers (right plot). Results are evaluated for an ideal turbojet with $\gamma = 1.3$, $Q_R/(c_pT_a) = 200$, $\eta_b = 1$, and $a_a = 300$ m/s.

the combustor and $B\dot{m}_a$ representing the air flow rate associated with the bypass stream. The bypass ratio $B$ is large in engines for commercial applications ($B \sim 10 - 12$) and much smaller ($B \sim 2 - 3$) in compact turbofans for military applications, the latter engines typically incorporating an afterburner. The core engine of a turbofan is basically a turbojet equipped with two turbines, one to move the compressor and one to move the fan that compresses the bypass stream, so that Eq. (73) must be replaced with

$$\mathcal{P}_t = \mathcal{P}_c + \mathcal{P}_f,$$

where $\mathcal{P}_t$ is the combined power of both turbines and $\mathcal{P}_f$ is the power needed to drive the fan.

Figure 15: Schematic view of a turbofan with two separate exhaust streams.

There are designs where the hot and cold streams mix upstream from the nozzle, leading to a single-exhaust jet engine. In the design of Fig. 15, however, the cold and hot streams remain separate up to the exit section. Because of the presence of distinct exhaust streams with speeds $u_e$ and $u_{ef}$, the resulting thrust equation (assuming $p_e = p_a$) takes the form

$$\mathcal{T} = (\dot{m}_a + \dot{m}_f)u_e + B\dot{m}_au_{ef} - \dot{m}_a(1 + B)u,$$

(81)
with corresponding specific thrust and TSFC given by

$$\frac{T}{m_a} = (1 + f)u_e + Bu_{ef} - (1 + B)u \quad \text{and} \quad \text{TSFC} = \frac{f}{(1 + f)u_e + Bu_{ef} - (1 + B)u}. \quad (82)$$

### 7.1 Performance analysis

As seen in (82), the turbofan performance depends on $f$, $u_e$, and $u_{ef}$. The evaluation of these three quantities, involving the fan pressure ratio $P_{rf}$ and the bypass ratio $B$ as additional inputs, parallels that developed above for the turbojet. In particular, since the analysis of the hot stream evolution across the core engine is identical to that of the turbojet, the fuel-to-air mass ratio $f$ and the exit speed for the hot jet $u_e$ are given by (66) and (72), respectively. However, the temperature jump across the turbine $T_{0s}/T_{0t}$, appearing in (72), is not given by the turbojet expression (75) stemming from the power balance (73). The value of $T_{0s}/T_{0t}$ for a turbofan is determined instead by the condition (80), as explained below.

The enthalpy-entropy evolution of the hot stream is that given on the right-hand side of Fig. 11. The corresponding evolution of the cold stream, including the effects of the diffuser, fan, and fan nozzle are represented in Fig. 16, with indication of the idealized states $7s$ and $8s$ used in the definitions of the fan and fan nozzle adiabatic efficiencies $\eta_f$ and $\eta_{fn}$.

![Figure 16: Enthalpy-entropy variation for the bypass stream.](image)

Across the fan, with shaft power $P_f$, energy conservation provides

$$P_f = B\dot{m}_a(h_{0f} - h_{0s}). \quad (83)$$

On the other hand, the temperature jump across the fan is related to the fan pressure ratio $P_{rf} = p_{0r}/p_{0s}$ through the fan adiabatic efficiency

$$\eta_f = \frac{h_{7s} - h_2}{h_7 - h_2} = \frac{P_{rf}^{\gamma - 1} - 1}{T_{0r}/T_{0s} - 1} \quad \Rightarrow \quad \frac{T_{0r}}{T_{0s}} = 1 + \frac{P_{rf}^{\gamma - 1} - 1}{\eta_f}. \quad (84)$$

Similarly, the analysis of the fan nozzle involves the energy equation

$$h_7 = h_{0r} = h_{0s} = h_8 + \frac{u_{ef}^2}{2} \quad \Rightarrow \quad h_7 - h_8 = \frac{u_{ef}^2}{2} \quad (85)$$
along with the definition of the corresponding adiabatic efficiency

$$\eta_{fn} = \frac{h_7 - h_8}{h_7} = \frac{u_{ef}^2/2}{h_7[1 - (p_a/p_T)^{\gamma-1}]} \rightarrow \frac{u_{ef}^2}{2} = \frac{\eta_{fn}a_a^2}{\gamma - 1} \left( \frac{T_{0\gamma}}{T_{02}} \right) \left( \frac{T_{02}}{T_a} \right) \left[ 1 - \frac{1}{\left( \frac{p_{0\gamma} p_{02}}{p_{0a} p_{02}} \right)^{\gamma-1}} \right], \quad (86)$$

the latter written with use of \( h_7 = h_{0\gamma} = c_p T_{0\gamma}, \) \( p_{0\gamma} = p_{0\gamma}, \) and \( c_p T_a = a_a^2/\gamma - 1 \). Substitution of (60), (61), (84), and \( P_{r_f} = p_{0\gamma}/p_{02} \) finally yields

$$\frac{u_{ef}^2}{2} = \frac{\eta_{fn}a_a^2}{\gamma - 1} \left( 1 + \frac{P_{r_f}^{\gamma-1} - 1}{\eta_f} \right) \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \left[ 1 - \frac{1}{P_{r_f}^{\gamma-1} \left( 1 + \eta_f^{\gamma-1} M^2 \right)} \right] \quad (87)$$

for the exit speed of the bypass stream.

To close the solution we need the temperature jump across the turbine \( T_{0\gamma}/T_{04} \), necessary in evaluating \( u_e \) from (72). Using (62), (67) and (83) to evaluate the power balance (80) gives

$$(1 + f)(T_{04} - T_{05}) = (T_{03} - T_{02}) + B(T_{0\gamma} - T_{02}) \rightarrow \frac{T_{05}}{T_{04}} = 1 - \frac{\left( \frac{T_{0\gamma}}{T_{02}} - 1 \right) + B \left( \frac{T_{0\gamma}}{T_{02}} - 1 \right)}{(1 + f)T_{04}/T_a} \left( \frac{T_{02}}{T_a} \right) \quad (88)$$

finally yielding

$$\frac{T_{05}}{T_{04}} = 1 - \left[ \frac{1}{\eta_c} \left( P_{r_c}^{\gamma-1} - 1 \right) + \frac{B}{\eta_f} \left( P_{r_f}^{\gamma-1} - 1 \right) \right] \frac{\left( 1 + \gamma-1 M^2 \right)}{(1 + f)(T_{04}/T_a)} \quad (89)$$

upon substitution of (60), (64), and (84).

### 7.2 Turbofan performance

To evaluate the specific thrust and the TSFC from (82) we use (66) to compute \( f \), (72) and (89) to compute \( u_e \), and (87) to compute \( u_{ef} \). The influence of the bypass ratio and the fan pressure ratio on the performance of the system is shown in Fig. 17 for an ideal turbofan flying at \( M = 0.8 \) with \( P_{r_c} = 20 \) (all other parameters in the evaluation are those of the turbojet analyzed in Fig. 13). As can be seen, for these subsonic conditions turbofans with a large bypass ratio are an attractive option with reduced fuel consumption and increased specific thrust. In reality, the value of \( B \) is limited by the structural weight and aerodynamic drag of the nacelle that is holding the engine as well as by compressibility effects affecting the flow near the tip of the fan blades.

### 8 Turboprop and turboshaft engines

Figure 18 shows a schematic view of a turboprop. Just like the turbofan, the turboprop uses as core engine a turbojet with two turbines, namely, a “compressor” turbine connected through a shaft to the compressor of the gas generator and a “power” turbine, mechanically independent of the gas-generator rotor elements, which is used to move a large external propeller. The amount of air that flows through the propeller is 25-30 times larger than that flowing through the gas generator, so in some sense a turboprop is a high-bypass-ratio engine. Because of the absence of a surrounding duct, near-tip compressibility effects in propellers are more pronounced than those found in turbofans. For that reason, turboprops are suited for subsonic flight at only moderately large Mach numbers \( M \lesssim 0.7 \).
Figure 17: Variation of the specific thrust and TSFC with the bypass ratio $B$ for an ideal turbofan with $P_r = 20$, $M = 0.8$, $\gamma = 1.3$, $Q_R/(c_pT_a) = 200$, $\eta_b = 1$, and $a_o = 300$ m/s.

Figure 18: Schematic view of a turboprop with free-turbine configuration.

In general, the thrust of a turboprop can be expected to have comparable contributions coming from the propeller and from the hot-exhaust jet. A turboshaft is a turboprop in which the hot gases are expanded through the power turbine to a near-ambient pressure, so that the exhaust velocity gives a negligible contribution to the thrust. Because of the extra weight of the power turbine, the gearbox (necessary to guarantee that the propeller and the engine operate at optimum rotational speeds), and the large propeller, with its accompanying pitch-control mechanism, turboprops are typically about 50% heavier than conventional turbojets with the same gas-generator size. The increased performance at takeoff and low flight speeds often compensate for this additional weight.

8.1 Performance analysis

The evolution of the gas through the turboprop of Fig. 18 is represented in the $h - s$ diagram of Fig. 19, including a detailed view of the last stages ($5 \rightarrow 6 \rightarrow 7 = e$), corresponding to the flow through the power turbine and the nozzle. The description of the evolution of the gas upstream from the power turbine (i.e. $a \rightarrow 5$) is identical to that described above in sections 6.1.1–6.1.4.

Assuming that the exit pressure $p_e$ is equal to the ambient pressure, the total thrust provided by the
The power-turbine power $P_{pt}$ and the exhaust speed $u_e$ are not independent, as shown on the right-hand side of Fig. 19. As can be seen, the energy available in the gas stream, measured by the difference $\Delta h = h_5 - h^*$ between the enthalpy $h_5$ behind the compressor turbine and the enthalpy $h^*$ corresponding to an isentropic expansion from $p_5$ to the ambient pressure $p_a$, is partly employed to move the propeller and partly employed to accelerate the gas stream through the nozzle. In the following, $\alpha \Delta h$ will denote the fraction of the available enthalpy used in the power turbine, with the remaining enthalpy $(1 - \alpha) \Delta h$ correspondingly used to accelerate the stream. The extreme cases $\alpha = 1$ and $\alpha = 0$ represent the turboshaft and the turbojet, respectively.

\[
\frac{T}{m_a} = (1 + f)u_e - u + \eta_{pr} \eta_g P_{pt},
\]

(91)

Substituting (61), (68) and the known pressure ratios $P_{rc} = p_{o3}/p_{o2}$ and $r_c = p_{o4}/p_{o3}$ finally gives

\[
\frac{\Delta h}{h_a} = \frac{T_{05} T_04}{T_{04} T_a} \left[ 1 - \frac{1}{\left( \frac{p_{o5}}{p_{o3}} \right)^{\frac{\gamma - 1}{\gamma}}} \right] = \frac{T_{05} T_04}{T_{04} T_a} \left[ 1 - \frac{1}{\left( \frac{p_{o5}}{p_{o4}} \frac{p_{o4}}{p_{o3}} \frac{p_{o2}}{p_{o3}} \right)^{\frac{\gamma - 1}{\gamma}}} \right],
\]

(92)

Substituting (61), (68) and the known pressure ratios $P_{rc} = p_{o3}/p_{o2}$ and $r_c = p_{o4}/p_{o3}$ finally gives

\[
\frac{\Delta h}{h_a} = \frac{T_{05} T_04}{T_{04} T_a} \left[ 1 - \frac{1}{\left( 1 - \frac{T_{05}/T_{04}}{\eta_t} \right)^{\frac{\gamma - 1}{\gamma}}} \frac{1}{P_{rc}^{\frac{\gamma - 1}{\gamma}} \left( 1 + \eta_{d}^{\frac{\gamma - 1}{2}} M^2 \right)} \right],
\]

(93)

with $T_{05}/T_{04}$ obtained from (75).
The analysis continues by considering the flow across the power turbine, whose energy balance reduces to

\[ P_{pt} = (\dot{m}_a + \dot{m}_f)(h_{05} - h_{06}). \]  

(94)

The adiabatic efficiency of the power turbine can be written as

\[ \eta_{pt} = \frac{h_5 - h_6}{h_5 - h_{6s}}. \]  

(95)

Since \( h_5 - h_{6s} = \alpha \Delta h \) and \( h_5 - h_6 = h_{05} - h_{06} \), it follows that \( h_{05} - h_{06} = \eta_{pt} \alpha \Delta h \), leading to

\[ \frac{P_{pt}}{\dot{m}_a} = (1 + f)\eta_{pt} \alpha \Delta h \]  

(96)

upon substitution into (94).

Similarly, the energy balance across the nozzle gives

\[ h_{06} = h_{07} = h_7 + \frac{u_e^2}{2} \rightarrow h_6 - h_7 = \frac{u_e^2}{2}. \]  

(97)

Since the enthalpy difference is \( h_6 - h_7 = \eta_n (1 - \alpha) \Delta h \) in terms of the nozzle adiabatic efficiency

\[ \eta_n = \frac{h_6 - h_7}{h_6 - h_{7s}} \simeq \frac{h_6 - h_7}{(1 - \alpha) \Delta h}. \]  

(98)

it follows from the last equation in (97) that

\[ u_e = \sqrt{2\eta_n (1 - \alpha) \Delta h} \]  

(99)

Substituting (96) and (99) into (91) provides

\[ \frac{T}{\dot{m}_a} = (1 + f)\sqrt{2\eta_n (1 - \alpha) \Delta h} - u + (1 + f)\eta_{pr} \eta_g \eta_{pt} \alpha \frac{\Delta h}{u} \]  

(100)

for the turboprop specific thrust, with \( \Delta h \) evaluated from (93) supplemented with (75).

### 8.2 Maximum specific thrust

For given values of the flight speed \( u \), fuel-to-air ratio \( f \), available enthalpy \( \Delta h \), and efficiencies \( \eta_n \) and \( \eta_{pr} \eta_g \eta_{pt} \), the optimum value \( \alpha_{OPT} \) of \( \alpha \) that results in the maximum specific thrust \( (T/\dot{m}_a)_{max} \) can be obtained by straightforward differentiation of (100) to give

\[ \frac{\partial (T/\dot{m}_a)}{\partial \alpha} = 0 \quad \rightarrow \quad \alpha_{OPT} = 1 - \frac{\eta_n}{(\eta_{pr} \eta_g \eta_{pt})^2} \frac{u^2 / 2}{\Delta h}, \]  

(101)

giving

\[ \left( \frac{T}{\dot{m}_a} \right)_{max} = u \left[ (1 + f)\eta_{pr} \eta_g \eta_{pt} \frac{\Delta h}{u^2} - \left( 1 - \frac{(1 + f)\eta_n}{2\eta_{pr} \eta_g \eta_{pt}} \right) \right] \]  

(102)

upon substitution of \( \alpha = \alpha_{OPT} \) into (100). Values of \( \alpha < \alpha_{OPT} \) (a faster exhaust stream and a propeller with smaller thrust power) or \( \alpha > \alpha_{OPT} \) (a slower exhaust stream and a propeller with higher thrust power) result in smaller values of the thrust coefficient.