Problem 1: A landing aircraft rolling down the runway at $u = 50 \text{ m/s}$ has an idling turbofan that consumes air at a rate $\dot{m}_a = \dot{m}_{ac} + \dot{m}_{ah} = 100 \text{ kg/s}$ and produces exhaust streams with velocities (relative to the aircraft) $u_{ec} = 100 \text{ m/s}$ and $u_{eh} = 150 \text{ m/s}$. For the analysis, consider that the bypass ratio of the turbofan is $B = \dot{m}_{ac}/\dot{m}_{ah} = 6$ and that the pressure at the exit equals the ambient value.

a) What is the forward thrust of the turbofan during landing? Simplify the expression for $f = \dot{m}_f/\dot{m}_{ah} \ll 1$.

b) What is the magnitude and direction (i.e., forward or reverse) of the thrust if the bypass exhaust is deflected $90^\circ$ without affecting either the air mass flows or the magnitudes of the exhaust velocities?

c) Compute the additional deflection $\alpha$ needed to get zero net thrust once the aircraft has come to a stop. For the new flow conditions, assume that the bypass ratio $B = 6$ and the velocity ratio $u_{ec}^*/u_{eh}^* = 2/3$ remain equal to those found while the aircraft is moving.

Solution: For steady flow with $p_e = p_a$ the $x$-component of the momentum equation

$$\left[ \frac{d}{dt} \int_{V_e} \rho v dV + \int_{\Sigma_e} \rho \mathbf{v} \cdot \mathbf{n} d\sigma \right]_x = -\int_{\Sigma_e} (p - p_a) \mathbf{n} d\sigma + T$$

reduces to

$$\dot{m}_{eh} u_{eh} + \dot{m}_{ec} u_{ec} - \dot{m}_a u = T,$$

which can be alternatively written by introduction of $\dot{m}_{eh} = \dot{m}_{ah} + \dot{m}_f$, $\dot{m}_{ec} = \dot{m}_{ac}$, and $f = \dot{m}_f/\dot{m}_{ah}$ in the form

$$T = \dot{m}_{ah} (1 + f) u_{eh} + \dot{m}_{ac} u_{ec} - (\dot{m}_{ah} + \dot{m}_{ac}) u.$$

The bypass ratio of the turbofan is defined as $B = \dot{m}_{ac}/\dot{m}_{ah}$, so by dividing the above expression by $\dot{m}_{ah}$ we obtain

$$\frac{T}{\dot{m}_{ah}} = (1 + f) u_{eh} + Bu_{ec} - (1 + B) u,$$

which further simplifies to

$$\frac{T}{\dot{m}_{ah}} = u_{eh} + Bu_{ec} - (B + 1) u.$$
in cases where \( f \ll 1 \). Evaluating this final expression for the conditions of the problem yields \( T \simeq 5714 \text{ N} \) (note that \( \dot{m}_{aeH} = \dot{m}_a/(1 + B) \simeq 14.29 \text{ kg/s} \)).

For part b), the bypass exhaust does not contribute to the momentum in the \( x \) direction, so that

\[
T = \dot{m}_{aeH}(1 + f)u_{eH} - (\dot{m}_{aeH} + \dot{m}_{ac})u,
\]

thereby yielding

\[
\frac{T}{\dot{m}_{aeH}} = u_{eH} - (B + 1)u \simeq -2857\text{ N}
\]

when the simplification \( f \ll 1 \) is incorporated. As can be seen, the resulting force is negative and therefore tends to slow down the plane.

For part c), the aircraft has also come to stop, so that the simplified momentum conservation equation gives in this case

\[
\frac{T}{\dot{m}_{aeH}} = u_{eH} - Bu_{ec} \sin \alpha.
\]

Equating the thrust to zero and solving for \( \alpha \) finally yields

\[
\sin \alpha = \frac{1}{Bu_{ec}/u_{eH}} = 1/4 \quad \rightarrow \quad \alpha \simeq 14.48^\circ.
\]
Problem 2: Using the expression derived in problem 1a for the thrust of a turbofan (before simplification), show that the associated thrust specific fuel consumption is

\[\text{TSFC} = \frac{\dot{m}_f}{T} = \frac{f}{(1 + f)u_{eH} + Bu_{ec} - (1 + B)u},\]

where \(f = \dot{m}_f/\dot{m}_{aH}\), while the overall efficiency becomes

\[\eta_o = \frac{T u}{\dot{m}_f Q R} = \frac{(1 + f)u_{eH} + Bu_{ec} - (1 + B)u}{(1 + f)u_{eH} + Bu_{ec} - (1 + B)u}.\]

Introduce the thrust-averaged exhaust velocity

\[\bar{u}_e = \frac{\dot{m}_{aH}(1 + f)u_{eH} + \dot{m}_{ac}u_{ec}}{\dot{m}_{aH}(1 + f) + \dot{m}_{ac}},\]

to derive the approximate result

\[\eta_o \simeq (1 + B)(\bar{u}_e - u)u \frac{f Q R}{u},\]

after using the condition \(f \ll 1\). For a given bypass ratio \(B\) and a given exhaust velocity \(\bar{u}_e\) show that the overall efficiency is maximized for \(u = \bar{u}_e/2\).

Solution: The thrust specific fuel consumption is defined as TSFC = \(\dot{m}_f/T\), so that

\[\text{TSFC} = \frac{\dot{m}_f}{\dot{m}_{aH}(1 + f)u_{eH} + \dot{m}_{ac}u_{ec} - (\dot{m}_{ac} + \dot{m}_{aH})u} = \frac{f}{(1 + f)u_{eH} + Bu_{ec} - (1 + B)u},\]

where we have used \(f = \dot{m}_f/\dot{m}_{aH}\) and \(B = \dot{m}_{ac}/\dot{m}_{aH}\). On the other hand, the overall efficiency is

\[\eta_o = \frac{T u}{\dot{m}_f Q R} = \frac{u}{\text{TSFC} Q R} = \frac{(1 + f)u_{eH} + Bu_{ec} - (1 + B)u}{f Q R} u.\]

Introducing the thrust-average velocity

\[\bar{u}_e = \frac{\dot{m}_{aH}(1 + f)u_{eH} + \dot{m}_{ac}u_{ec}}{\dot{m}_{aH}(1 + f) + \dot{m}_{ac}} = \frac{(1 + f)u_{eH} + Bu_{ec}}{1 + f + B}\]

provides

\[\eta_o = \frac{(1 + f + B)\bar{u}_e - (1 + B)u}{f Q R} u \simeq (1 + B)(\bar{u}_e - u)u \frac{f Q R}{u},\]

where the latter expression follows from the approximation \(f \ll 1\).

For a given bypass ratio \(B\) and a given exhaust velocity \(\bar{u}_e\) the overall efficiency depends on \(u/\bar{u}_e\) according to

\[\eta_o = \frac{(1 + B)\bar{u}_e^2}{f Q R} \left(1 - \frac{u}{\bar{u}_e}\right) \frac{u}{\bar{u}_e}.\]

To obtain the maximum, we take the derivative of the above expression with respect to \(u/\bar{u}_e\) and equate the result to zero to give \(u/\bar{u}_e = 1/2\).
Problem 3: Use Brequet’s range equation

\[ \frac{m_{\text{payload}}}{m_o} = \exp \left( -\frac{gs_{\text{max}}}{\eta_o Q R (L/D)} \right) - \frac{m_{\text{empty}}}{m_o}, \]

where \( m_o \) is the maximum aircraft mass at takeoff. Use the above equation to represent the payload vs range diagram \( \left( \frac{m_{\text{payload}}}{m_o} - s_{\text{max}} \right) \) with \( s_{\text{max}} \) in km for an aircraft with \( L/D = 15 \) and \( m_{\text{empty}}/m_o = 0.7 \) whose jet engine has an overall efficiency \( \eta_o = 0.6 \). Use the value \( Q R = 45,000 \) kJ/kg, typical of hydrocarbon fuels, in the evaluation.

Solution:

\[ s = \frac{\eta_o Q R (L/D)}{g} \ln \left( \frac{m_{\text{init}}}{m_{\text{final}}} \right) \Rightarrow \frac{gs}{\eta_o Q R (L/D)} = \ln \left( \frac{m_{\text{init}}}{m_{\text{final}}} \right) \Rightarrow \frac{m_{\text{init}}}{m_{\text{final}}} = \exp \left( \frac{gs}{\eta_o Q R (L/D)} \right) \]

The initial mass \( m_{\text{init}} = m_{\text{empty}} + m_{\text{payload}} + m_{\text{fuel}} \) is limited by the airplane structure and the power of the engines to a maximum value at takeoff \( m_{\text{init}} = m_0 \), while the final airplane mass after depleting all of the fuel is \( m_{\text{final}} = m_{\text{empty}} + m_{\text{payload}} \), thereby yielding

\[ \frac{m_{\text{payload}}}{m_o} = \exp \left( -\frac{gs_{\text{max}}}{\eta_o Q R (L/D)} \right) - \frac{m_{\text{empty}}}{m_o}. \]

The above expression is used below to plot the payload vs range diagram for the airplane defined in the problem.