**Problem 1:** A planar supersonic jet with \( M_1 = 2 \) impinges against a wedge of semi-angle \( \delta = 17^\circ \), generating a complex flow field structure that includes initially an oblique shock followed by a Prandtl-Meyer expansion, as shown in the figure.

- Determine the Mach number behind the oblique shock \( M_2 \).
- Compute the pressure jump \( p_2/p_a \) across the shock.
- Obtain the Mach number immediately behind the expansion \( M_3 \).
- Find the temperature behind the expansion \( T_3 \), giving the result in the form \( T_3/T_1 \), with \( T_1 \) representing the initial jet temperature.
- Calculate the deflection of the jet, defined by the angle \( \alpha \) in the figure.

**Solution:**

- For \( \delta = 17^\circ \) and \( M_1 = 2 \), we obtain \( \beta = 48^\circ \) from the \( \beta - \delta \) diagram, so that \( M_{1n} = M_1 \sin(\beta) = 1.468 \). The Rankine-Hugoniot jump conditions then provide \( M_{2n} = 0.706 \) and \( p_2/p_a = 2.41 \). Finally, \( M_2 = M_{2n}/\sin(\beta - \delta) = 1.371 \). Alternatively, these values can be obtained directly from the oblique-shock diagram of pressure jump vs incident Mach number.

- We may use the conditions \( p_{02} = p_{03} \) and \( p_3 = p_a \) to write

\[
\frac{p_{03}}{p_3} = \left(1 + \frac{\gamma - 1}{2} M_3^2\right)^{\frac{\gamma}{\gamma - 1}} \quad \frac{p_{02}}{p_2} = \frac{p_2}{p_a} = \left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{\frac{\gamma}{\gamma - 1}} \frac{p_2}{p_a},
\]

which can be solved to give

\[
M_3 = \left\{ \frac{2}{\gamma - 1} \left[ \left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{\frac{\gamma}{\gamma - 1}} - 1 \right] \right\}^{1/2} = 1.961
\]
• The stagnation temperature is conserved between 1 and 3, so that

\[ T_{01} = T_{03} \Rightarrow \frac{T_3}{T_1} = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_3^2} = 1.0175 \]

• The value of \( \theta \) can be found using the expression \( \theta = \nu(M_3) - \nu(M_2) = 25.29^\circ - 8.14^\circ = 17.15^\circ \), so that \( \alpha = \delta + \theta = 34.15^\circ \)
**Problem 2:** In analyzing supersonic aerodynamic intakes the flow may be assumed in the first approximation to be isentropic, so that all stagnation properties are conserved. In reality, the compression takes place through shock waves that increase the entropy and reduce the stagnation pressure. As a simplified model to evaluate this effect, consider the planar configuration shown in the figure, including a wall with an intermediate straight section with deflection angle $\delta$ that gives rise to the appearance of a system of oblique shocks. Assume that for cruise conditions the intake is designed for the reflected shock wave to impinge at the downstream wall corner, so that no further shocks or expansions appear downstream. For $M_1 = 3$, determine in the three cases $\delta = (10^\circ, 15^\circ, 20^\circ)$:

1. The downstream Mach number $M_3$.
2. The relative decrease in stagnation pressure $(p_{0_1} - p_{0_3})/p_{0_1}$.
3. The ratio of stagnation temperatures $T_{0_3}/T_{0_1}$.
4. The increase in entropy $(s_3 - s_1)/c_v$. Discuss the accuracy of the isentropic-flow approximation.

Obtain also the maximum value of $\delta$ for which the flow configuration shown in the figure may exist.

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**Solution:**

1. Using the oblique-shock diagram of pressure jump vs incident Mach number we obtain

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$M_2$</th>
<th>$p_2/p_1$</th>
<th>$M_3$</th>
<th>$p_3/p_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^\circ$</td>
<td>2.50</td>
<td>2.10</td>
<td>2.10</td>
<td>1.85</td>
</tr>
<tr>
<td>$15^\circ$</td>
<td>2.25</td>
<td>2.80</td>
<td>1.65</td>
<td>2.30</td>
</tr>
<tr>
<td>$20^\circ$</td>
<td>2.00</td>
<td>3.75</td>
<td>1.20</td>
<td>2.80</td>
</tr>
</tbody>
</table>

2. Since $p_0/p = (1 + \frac{\gamma - 1}{2}M_1^2)^{\gamma/(\gamma - 1)}$, the relative decrease in stagnation pressure is given by

   $$(p_{0_1} - p_{0_3})/p_{0_1} = 1 - \frac{p_3}{p_2} \frac{p_2}{p_1} \left( \frac{1 + \frac{\gamma - 1}{2}M_3^2}{1 + \frac{\gamma - 1}{2}M_1^2} \right)^{\gamma/(\gamma - 1)},$$

   so that

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$(p_{0_1} - p_{0_3})/p_{0_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^\circ$</td>
<td>0.0328</td>
</tr>
<tr>
<td>$15^\circ$</td>
<td>0.1972</td>
</tr>
<tr>
<td>$20^\circ$</td>
<td>0.3070</td>
</tr>
</tbody>
</table>

3. The stagnation temperature is conserved across a shock-wave, i.e. $T_{0_3}/T_{0_1} = 1$. 

4. The increase in entropy can be related to the jump in stagnation pressure according to

\[
\frac{(s_3 - s_1)}{c_v} = \ln \left[ \frac{p_3/p_1}{(p_3/p_1)^{\gamma/2}} \right] = (\gamma - 1) \ln \left( \frac{p_0}{p_0} \right).
\]

<table>
<thead>
<tr>
<th>(\delta)</th>
<th>(\frac{(s_3 - s_1)}{c_v})</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td>(1.33 \times 10^{-2})</td>
</tr>
<tr>
<td>15°</td>
<td>(8.80 \times 10^{-2})</td>
</tr>
<tr>
<td>20°</td>
<td>0.1466</td>
</tr>
</tbody>
</table>

The assumption of isentropic flow is accurate for sufficiently small values of \(\delta\), associated with weak shocks.

The maximum deflection \(\delta \approx 21^\circ\) for which a regular reflection exists is associated with the existence of the second oblique shock. It can be determined by iteration or (almost) directly from the oblique-shock diagram of pressure jump vs incident Mach number.
Problem 3: The picture shows a Lockheed SR-71, also known as Blackbird, a long-range, Mach 3+ strategic reconnaissance aircraft developed in the sixties.

At take-off, the corresponding nozzle flow is over-expanded, so that an oblique shock wave appears at the nozzle exit, followed by other shock waves and expansions, causing the jet to exhibit a characteristic diamond-like pattern. A simplified analysis of the associated flow can be based on the planar configuration shown in the figure below.

For known values of $p_1 = 0.5p_a$ and $M_1 = 3.2$, determine:

1. The value $M_2$ of the Mach number behind the first shock wave. **Solution:** $M_2 = 2.7$, $\delta = \alpha_1 = 9^\circ$.

2. The values of $M_3$ and pressure $p_3$ behind the second shock wave, giving the pressure in the form $p_3/p_a$. **Solution:** $M_3 = 2.3$ and $p_3/p_2 = p_3/p_a = 1.8$.

3. The value of $M_4$ behind the expansion. **Solution:** from the condition $p_0_1 = p_0_4$ with $p_4 = p_a$ it follows that $M_4 = 2.68$.

4. The angles $\alpha_1$ and $\alpha_2$ defining the jet surface, which is a tangential discontinuity (represented by a dashed line in the figure). **Solution:** $\alpha_2 = \nu(M_4) - \nu(M_3) = 9^\circ$.

5. The temperature in the different regions referred to that at the jet exit (i.e. $T_2/T_1$, $T_3/T_1$, and $T_4/T_1$). **Solution:** $T_2/T_1 = 1.23$, $T_3/T_1 = 1.46$, and $T_4/T_1 = 1.234$.
Problem 4: The supersonic intake shown in the figure, whose inlet-to-outlet area ratio is \( A_2/A_1 = 0.3985 \), is part of the propulsion system of a fighter aircraft designed to fly at a cruise Mach number \( M_1 = 3 \). Behind the aerodynamic compressor, assumed to be isentropic, there exists a release gate of variable area \( A_4 \) open to the atmosphere, whose objective is to control the air flow that circulates towards the engine.

- Determine the values of \( p_2/p_1 \) and \( M_2 \).
- Obtain the values of \( p_3/p_1 = p_4/p_1 \) at the release gate, as well as the Mach numbers \( M_3 \) and \( M_4 \) and the deflection angle \( \alpha \).
- Give an expression for the fraction of mass flow rate that is released back to the atmosphere, \((\rho_4 v_4 A_4 \sin \alpha)/(\rho_2 v_2 A_2)\), as a function of \( A_4/A_2 \). Determine the value of \( A_4/A_2 \) for which this fraction is 10%.
- An oblique shock is seen to form at the rear edge of the opening. Obtain the values of \( M_5 \) and \( p_5/p_1 \) found immediately downstream.

![Diagram](image)

Solution:

- We assume that the evolution from 1 to 2 is isentropic and steady. Using the condition of equal mass flow rate \( \dot{m}_1 = \dot{m}_2 \) provides

\[
\dot{m}_1 = \dot{m}_2 \Rightarrow \frac{\rho_1 a_1 M_1}{\rho_2 a_2 M_2} = \frac{A_2}{A_1} \Rightarrow \frac{M_1}{M_2} = \frac{1 + \frac{\gamma-1}{2} M_2^2}{1 + \frac{\gamma-1}{2} M_1^2} = \frac{A_2}{A_1},
\]

which can be solved to give \( M_2 = 2 \).\(^1\) Since \( p_0 = p_0_2 \), it follows that

\[
\frac{p_2}{p_1} = \frac{p_2/p_0_2}{p_1/p_0_1} = \left( \frac{1 + \frac{\gamma-1}{2} M_2^2}{1 + \frac{\gamma-1}{2} M_1^2} \right)^{-\gamma/(\gamma-1)} \approx 4.694.
\]

- Since \( p_2 > p_1 \), an oblique shock and an expansion wave appear when the two supersonic streams 1 and 2 meet at the rim of the gate. The flow undergoes the same deflection across the shock (\( \delta = \alpha \)) and across the expansion (\( \theta = \alpha \)), so that the two streams 3 and 4 run

\(^1\)Alternatively, since \( A_1^* = A_2^* \), we may write \( A_2^*/A_2 = (A_1^*/A_1)(A_1/A_2) = 0.5924 \), with \( A_1^*/A_1 = 0.236152 \) obtained from the table of steady isentropic flow for \( M_1 = 3 \). Using that same table for \( A_2^*/A_2 = 0.5924 \) provides \( M_2 = 2 \).
parallel to each other, separated by a tangential discontinuity (i.e. a vortex sheet), so that $p_3 = p_4$. To determine the solution we need to proceed by trial and error. For a given value of $\alpha$ we can use $\nu(M_4) = \alpha + \nu(M_2) = \alpha + 26.38$ to obtain $M_4$. Also, since $p_0 = p_1$, we may write

$$p_4 \left(1 + \frac{\gamma - 1}{2} M_4^2\right)^{\frac{1}{\gamma - 1}} = p_2 \left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{\frac{1}{\gamma - 1}} \rightarrow \frac{p_4}{p_1} = \left(\frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_4^2}\right)^{\frac{1}{\gamma - 1}} \frac{p_2}{p_1}.$$  

On the other hand, the pressure jump $p_3/p_1$ across the shock can be obtained from the oblique-shock diagram of pressure jump vs incident Mach number with $\delta = \alpha$ and $M_1 = 3$.

Results are given in the table below for two different values of $\alpha$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$p_3/p_1$</th>
<th>$M_4$</th>
<th>$p_4/p_2$</th>
<th>$p_4/p_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td>2.100</td>
<td>2.400</td>
<td>0.535</td>
<td>2.510</td>
</tr>
<tr>
<td>12°</td>
<td>2.350</td>
<td>2.460</td>
<td>0.488</td>
<td>2.280</td>
</tr>
</tbody>
</table>

The value of $\alpha$ appears to be slightly smaller than 12°. We shall take 12° for simplicity. The final solution is approximately given by $p_3/p_1 = p_4/p_1 \approx 2.3$, $M_3 \approx 2.4$ and $M_4 \approx 2.46$.

- The stagnation density and the stagnation speed of sound are conserved across the expansion wave, so that

$$\frac{\rho_4 v_4 A_4 \sin(\alpha)}{\rho_2 v_2 A_2} = \left(\frac{\rho_4 u_4 A_4}{\rho_2 u_2 A_2}\right) \frac{M_4 A_4 \sin(\alpha)}{M_2 A_2} = 0.138 \frac{A_4}{A_2}.$$  

For $\dot{m}_4 = \rho_4 u_4 A_4 \sin(\alpha) = 0.1 \dot{m}_2$ we obtain $A_4/A_2 = 0.1/0.138 = 0.725$.

- Using the values of $\alpha = 12^\circ$ and $M_4 = 2.46$, the values of $M_5 = 1.96$ and $p_5/p_4 \approx 2.1$ can be obtained from the oblique-shock diagram of pressure jump vs incident Mach number, yielding $p_5/p_1 = (p_5/p_4)(p_4/p_1) \approx 4.83$.  
