Problem 1: The supersonic intake shown in the figure, whose inlet-to-outlet area ratio is $A_2/A_1 = 0.3985$, is part of the propulsion system of a fighter aircraft designed to fly at a cruise Mach number $M_1 = 3$. Behind the aerodynamic compressor, assumed to be isentropic, there exists a release gate of variable area $A_4$ open to the atmosphere, whose objective is to control the air flow that circulates towards the engine.

- Determine the values of $p_2/p_1$ and $M_2$.
- Obtain the values of $p_3/p_1 = p_4/p_1$ at the release gate, as well as the Mach numbers $M_3$ and $M_4$ and the deflection angle $\alpha$.
- Give an expression for the fraction of mass flow rate that is released back to the atmosphere, \( \frac{\rho_4 v_4 A_4 \sin \alpha}{\rho_2 v_2 A_2} \), as a function of $A_4/A_2$. Determine the value of $A_4/A_2$ for which this fraction is 10%.
- An oblique shock is seen to form at the rear edge of the opening. Obtain the values of $M_5$ and $p_5/p_1$ found immediately downstream.

Solution:

- We assume that the evolution from 1 to 2 is isentropic and steady. Using the condition of equal mass flow rate $G_1 = G_2$ provides

\[
G_1 = G_2 \Rightarrow \frac{p_1 a_1 M_1}{p_2 a_2 M_2} = \frac{A_2}{A_1} \Rightarrow \frac{M_1}{M_2} \left( \frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_1^2} \right)^{\frac{\gamma + 1}{\gamma - 1}} = \frac{A_2}{A_1},
\]

which can be solved to give $M_2 = 2$ \(^1\). Since $p_0_1 = p_0_2$, it follows that

\[
\frac{p_2}{p_1} = \frac{p_2/p_0_2}{p_1/p_0_1} = \left( \frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_1^2} \right)^{-\gamma/(\gamma - 1)} \approx 4.694.
\]

\(^1\)Alternatively, since $A_1^* = A_2^*$, we may write $A_2^*/A_2 = (A_1^*/A_1)(A_1/A_2) = 0.5924$, with $A_1^*/A_1 = 0.236152$ obtained from the table of steady isentropic flow for $M_1 = 3$. Using that same table for $A_2^*/A_2 = 0.5924$ provides $M_2 = 2$. 
• Since \( p_2 > p_1 \), an oblique shock and an expansion wave appear when the two supersonic streams 1 and 2 meet at the rim of the gate. The flow undergoes the same deflection across the shock (\( \delta = \alpha \)) and across the expansion (\( \theta = \alpha \)), so that the two streams 3 and 4 run parallel to each other, separated by a tangential discontinuity (i.e. a vortex sheet), so that \( p_3 = p_4 \). To determine the solution we need to proceed by trial and error. For a given value of \( \alpha \) we can use \( \nu(M_4) = \alpha + \nu(M_2) = \alpha + 26.38 \) to obtain \( M_4 \). Also, since \( p_0^2 = p_0^4 \), we may write

\[
p_4 \left( 1 + \frac{\gamma - 1}{2} M_4^2 \right)^{\frac{\gamma-1}{\gamma}} = p_2 \left( 1 + \frac{\gamma - 1}{2} M_2^2 \right)^{\frac{\gamma-1}{\gamma}} \to \frac{p_4}{p_1} = \left( 1 + \frac{2 - 1}{2} M_2^2 \right)^{\frac{\gamma-1}{\gamma}} \frac{p_2}{p_1}.
\]

On the other hand, the pressure jump \( p_3/p_1 \) across the shock can be obtained from the oblique-shock diagram of pressure jump vs incident Mach number with \( \delta = \alpha \) and \( M_1 = 3 \). Results are given in the table below for two different values of \( \alpha \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( p_3/p_1 )</th>
<th>( M_4 )</th>
<th>( p_4/p_2 )</th>
<th>( p_4/p_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td>2.100</td>
<td>2.400</td>
<td>0.535</td>
<td>2.510</td>
</tr>
<tr>
<td>12°</td>
<td>2.350</td>
<td>2.460</td>
<td>0.488</td>
<td>2.280</td>
</tr>
</tbody>
</table>

The value of \( \alpha \) appears to be slightly smaller than 12°. We shall take 12° for simplicity. The final solution is approximately given by \( p_3/p_1 = p_4/p_1 \approx 2.3 \), \( M_3 \approx 2.4 \) and \( M_4 \approx 2.46 \).

• The stagnation density and the stagnation speed of sound are conserved across the expansion wave, so that

\[
\frac{\rho_4 v_4 A_4 \sin(\alpha)}{\rho_2 v_2 A_2} = \frac{\rho_4}{\rho_2} \frac{a_4}{a_2} \frac{M_4 A_4 \sin(\alpha)}{M_2 A_2} = 0.138 \frac{A_4}{A_2}
\]

For \( G_4 = \rho_4 u_4 A_4 \sin \alpha = 0.1 G_2 \) we obtain \( A_4/A_2 = 0.1/0.138 = 0.725 \).

• Using the values of \( \alpha \) and \( M_4 \), the values of \( M_5 = 1.96 \) and \( p_5/p_4 = 2.46 \) can be obtained from the oblique-shock diagram of pressure jump vs incident Mach number, yielding \( p_5/p_1 = (p_5/p_4)(p_4/p_1) = 4.83 \).
Problem 2: The thrust of a high-altitude vehicle is provided by a solid-propellant rocket engine that discharges to the atmosphere through a convergent-divergent nozzle with exit-to-throat area ratio \( A_e/A_t \). When operating in the upper atmosphere, the ambient pressure is so small that in the first approximation we may assume \( p_a = 0 \).

- Under those conditions, determine the minimum value of \( A_e/A_t \) needed to avoid reverse flow near the nozzle rim.

During the reentry, the ambient pressure increases continuously, giving also increasing values of the ratio \( p_a/p_0 \), where \( p_0 \) is the constant stagnation pressure in the combustion chamber upstream from the nozzle entrance. For the value of \( A_e/A_t \) calculated previously, determine the value of \( p_a/p_0 \) for which:

- The nozzle discharges to the ambient as a supersonic jet without expansions or shocks.
- There exists a normal shock wave standing at the nozzle exit.
- The nozzle unblocks, so that the flow is subsonic everywhere along the nozzle, except at the throat, where \( M = 1 \).

Solution:

- With ambient pressure \( p_a = 0 \) we find at the nozzle exit a supersonic jet expanding to \( M = \infty \). Since the maximum deflection \( \theta \) to avoid reverse flow is 90°, we may write \( \theta = 90° = \nu(\infty) - \nu(M_e) \), which yields \( M_e = 2.56 \). Since \( A^* = A_t \) for the flow in the nozzle, we may use the tables for steady isentropic flow with \( M_e = 2.56 \) to determine \( A_t/A_e = A_e/A_e = 0.357 \), so that \( A_e/A_t \sim 2.8 \).

- If the nozzle discharges as a supersonic jet without shocks or expansions, the ratio of \( p_a/p_0 = p_e/p_0 \) is given by:
  \[
  p_a/p_0 = p_e/p_0 = \frac{1}{\left(1 + \frac{\gamma-1}{2} M_e^2 \right)^{\gamma-1}} = 0.0533.
  \]

- With a normal shock wave at the exit, the ratio of \( p_a/p_0 \) is given by
  \[
  \frac{p_a}{p_0} = \frac{p_a}{p_e} \frac{p_e}{p_0} = 0.399,
  \]
  where \( p_a/p_e = 7.48 \) is found from the normal-shock tables and \( p_e/p_0 \) is the value previously calculated.

- When the flow is subsonic everywhere except at the throat, where \( M = 1 \), the ratio of \( p_a/p_0 = p_e/p_0 = 0.969 \) is found directly from the tables of steady isentropic flow by looking up \( A_t/A_e = A_s/A_e = 0.357 \) along the subsonic branch (the resulting discharge Mach number is \( M_e = 0.212 \)).
Problem 3: An air container at pressure $p_e$ and density $\rho_e$ discharges to an open atmosphere at pressure $p_a < p_e$ through a convergent-divergent nozzle with exit-to-throat area ratio $A_e/A_t = 4$.

1. Obtain the values of $p_a/p_e$ for which (i) the flow is everywhere subsonic, except at the throat, where it is sonic, (ii) the nozzle discharge occurs as a supersonic jet with $p_e = p_a$, and (iii) a normal shock is found at the exit section.

2. For $p_a/p_e = 0.99$, obtain the values of the Mach number at the exit and at the throat $M_e$ and $M_t$. Determine the mass flow rate $G$, giving the result in the form $G/(\sqrt{\gamma p_e p_c A_e})$.

3. For $p_a/p_e = 0.1$, obtain the values of $M_e$ and $M_t$ as well as the mass flow rate $G/(\sqrt{\gamma p_e p_c A_e})$. An oblique shock is formed at the exit section. Determine the Mach number immediately downstream as well as the deflection angle.

4. For $p_a/p_e = 0.025$, obtain the values of $M_e$ and $M_t$ as well as the mass flow rate $G/(\sqrt{\gamma p_e p_c A_e})$. The expansion formed at the exit section is a Prandtl-Meyer expansion in the vicinity of the nozzle rim. Determine the Mach number found downstream as well as the deflection angle.

Solution:

1. For cases (i) and (ii) we have $p_e = p_a$ and $A_t = A^*$. Using the tables of steady isentropic flow with $A^*/A_e = A_t/A_e = 0.25$ yields two solutions. The subsonic discharge corresponds to case (i), for which we find $M_e = 0.146$ and $p_e/p_c = p_a/p_e = p_{cni}/p_c = 0.985$. The supersonic discharge corresponds to case (ii), for which we find $M_e = 2.94$ and $p_e/p_c = p_a/p_c = p_{SS}/p_c = 0.0298$. For conditions (iii), the pressure ratio is given by $p_a/p_c = (p_a/p_e)(p_e/p_c) = p_{SS}/p_c = 0.2955$, where $p_e/p_c$ is the value found in case (ii) and $p_a/p_e$ is found from the normal-shock tables with $M_e = 2.94$.

2. Since $p_a/p_e = 0.99 > p_{cni}/p_c$, the flow is subsonic everywhere. The exit pressure of the subsonic jet is $p_e = p_a$, so that the condition of conservation of stagnation pressure provides\(^2\)

\[
\frac{p_e}{p_0} = \frac{p_a}{p_e} = 0.99 = \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{-\frac{\gamma}{\gamma - 1}} \to M_e \simeq 0.12.
\]

The value of $G$ has to be conserved along the nozzle, so that

\[
\rho_e u_e A_e = \rho_t u_t A_t \to \frac{\rho_e}{\rho_e} \frac{u_e}{u_e} \frac{M_e}{M_t} = \frac{A_t}{A_e} \to M_e \left(1 + \frac{\gamma - 1}{2} M_t^2\right)^{\frac{\gamma + 1}{\gamma - 1}} \to M_t = 0.576.
\]

The value of $G$ expressed as $G/(\sqrt{\gamma p_e p_c A_e})$ is

\[
G/(\sqrt{\gamma p_e p_c A_e}) = M_e \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{-\frac{\gamma + 1}{2(\gamma - 1)}} \simeq 0.119.
\]

3. Since the value $p_a/p_e = 0.1$ is in the range $p_{SS}/p_c > p_a/p_c > p_{cni}/p_c$ an oblique shock appears at the exit. The conditions inside the nozzle are simply $M_e = 2.94$ and $M_t = 1$, so that

\[
G/(\sqrt{\gamma p_e p_c A_e}) = M_e \left(1 + \frac{\gamma - 1}{2} M_t^2\right)^{-\frac{\gamma + 1}{2(\gamma - 1)}} = 0.1447.
\]

\(^2\)The value of $M_t = 0.12$ can be alternatively found directly from the tables of steady isentropic flow with $p/p_0 = p_a/p_e = 0.99$. Also, since $A^*$ is conserved, we may write $A^*/A_t = (A^*/A_e)(A_e/A_t) = 0.20558 \times 4 = 0.822$, where $A^*/A_e$ is the value corresponding to $M_e = 0.12$ and $p/p_0 = 0.99$. Using $A^*/A_t = 0.822$ in the tables of steady isentropic flow gives $M_t \simeq 0.576$.\]
The shock pressure jump is \( \frac{p_a}{p_e} = (p_a/p_e)/(p_e/p_c) = 0.1/0.0298 = 3.356 \). Using this pressure ratio and \( M_e = 2.94 \) together with the oblique-shock diagram of pressure jump vs incident Mach number yields \( \delta = 22^\circ \) and \( M_2 \simeq 1.9 \).

4. Since \( p_a/p_e < p_{SJ}/p_c \) we find an isentropic expansion wave at the nozzle exit. The conditions inside the nozzle are simply \( M_e = 2.94 \) and \( M_t = 1 \), so that

\[
\frac{G}{\sqrt{\gamma p_c p_c A_e}} = M_e \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} = 0.1447.
\]

Conservation of stagnation pressure across the expansion provides

\[
p_{02} = p_e \rightarrow \frac{p_e}{p_a} = \left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{-\frac{\gamma}{\gamma-1}} = 40 \rightarrow M_2 = 3.056.
\]

Near the edge, the expansion is locally planar and the solution reduces to the Prandtl-Meyer expansion, whose deflection is obtained according to \( \delta = \nu(M_2) - \nu(M_e) \simeq 1.93^\circ \).