Problem 1: The thrust of a high-altitude vehicle is provided by a solid-propellant rocket engine that discharges to the atmosphere through a convergent-divergent nozzle with exit-to-throat area ratio $A_e/A_t$. When operating in the upper atmosphere, the ambient pressure is so small that in the first approximation we may assume $p_a = 0$.

- Under those conditions, determine the minimum value of $A_e/A_t$ needed to avoid reverse flow near the nozzle rim.

During the reentry, the ambient pressure increases continuously, giving also increasing values of the ratio $p_a/p_0$, where $p_0$ is the constant stagnation pressure in the combustion chamber upstream from the nozzle entrance. For the value of $A_e/A_t$ calculated previously, determine the value of $p_a/p_0$ for which:

- The nozzle discharges to the ambient as a supersonic jet without expansions or shocks.
- There exists a normal shock wave standing at the nozzle exit.
- The nozzle unblocks, so that the flow is subsonic everywhere along the nozzle, except at the throat, where $M = 1$.

Solution:

- With ambient pressure $p_a = 0$ we find at the nozzle exit a supersonic jet expanding to $M = \infty$. Since the maximum deflection $\theta$ to avoid reverse flow is $90^\circ$, we may write $\theta = 90^\circ = \nu(\infty) - \nu(M_e)$, which yields $M_e = 2.56$. Since $A^* = A_t$ for the flow in the nozzle, we may use the tables for steady isentropic flow with $M_e = 2.56$ to determine $A_t/A_e = A^* / A_e = 0.357$, so that $A_e/A_t \approx 2.8$.

- If the nozzle discharges as a supersonic jet without shocks or expansions, the ratio of $p_a/p_0 = p_e/p_0$ is given by:

$$
p_a/p_0 = p_e/p_0 = \frac{1}{\left(1 + \frac{\gamma-1}{2} M_e^2\right)^{\frac{\gamma}{\gamma-1}}} = 0.0533.
$$

- With a normal shock wave at the exit, the ratio of $p_a/p_0$ is given by

$$
p_a/p_0 = \frac{p_a}{p_0} \frac{p_e}{p_0} = 0.399,
$$

where $p_a/p_e = 7.48$ is found from the normal-shock tables and $p_e/p_0$ is the value previously calculated.

- When the flow is subsonic everywhere except at the throat, where $M = 1$, the ratio of $p_a/p_0 = p_e/p_0 = 0.969$ is found directly from the tables of steady isentropic flow by looking up $A_t/A_e = A^* / A_e = 0.357$ along the subsonic branch (the resulting discharge Mach number is $M_e = 0.212$).
Problem 2: An air container at pressure $p_a$ and density $\rho_a$ discharges to an open atmosphere at pressure $p_a < p_c$ through a convergent-divergent nozzle with exit-to-throat area ratio $A_e/A_t = 4$.

1. Obtain the values of $p_a/p_c$ for which (i) the flow is everywhere subsonic, except at the throat, where it is sonic, (ii) the nozzle discharge occurs as a supersonic jet with $p_e = p_a$, and (iii) a normal shock is found at the exit section.

2. For $p_a/p_c = 0.99$, obtain the values of the Mach number at the exit and at the throat $M_e$ and $M_t$. Determine the mass flow rate $\dot{m}$, giving the result in the form $\dot{m}/(\sqrt{\gamma p_c A_e})$.

3. For $p_a/p_c = 0.1$, obtain the values of $M_e$ and $M_t$ as well as the mass flow rate $\dot{m}/(\sqrt{\gamma p_c A_e})$. An oblique shock is formed at the exit section. Determine the Mach number immediately downstream as well as the deflection angle.

4. For $p_a/p_c = 0.025$, obtain the values of $M_e$ and $M_t$ as well as the mass flow rate $\dot{m}/(\sqrt{\gamma p_c A_e})$. The expansion formed at the exit section is a Prandtl-Meyer expansion in the vicinity of the nozzle rim. Determine the Mach number found downstream as well as the deflection angle.

Solution:

1. For cases (i) and (ii) we have $p_e = p_a$ and $A_t = A^*$. Using the tables of steady isentropic flow with $A^*/A_e = A_t/A_e = 0.25$ yields two solutions. The subsonic discharge corresponds to case (i), for which we find $M_e = 0.146$ and $p_e/p_c = p_a/p_c = p_{cm}/p_c = 0.985$. The supersonic discharge corresponds to case (ii), for which we find $M_e = 2.94$ and $p_e/p_c = p_a/p_c = p_{ss1}/p_c = 0.0298$. For conditions (iii), the pressure ratio is given by $p_a/p_c = (p_a/p_c)(p_e/p_c) = p_{ss}/p_c = 0.2955$, where $p_e/p_c$ is the value found in case (ii) and $p_a/p_c$ is found from the normal-shock tables with $M_e = 2.94$.

2. Since $p_a/p_c = 0.99 > p_{cm}/p_c$, the flow is subsonic everywhere. The exit pressure of the subsonic jet is $p_e = p_a$, so that the condition of conservation of stagnation pressure provides

$$\frac{p_e}{p_{0_e}} = \frac{p_a}{p_c} = 0.99 = \left(1 + \frac{\gamma - 1}{2}M_e^2\right)^{-\frac{\gamma}{\gamma - 1}} \rightarrow M_e \simeq 0.12.$$ 

The value of $\dot{m}$ has to be conserved along the nozzle, so that

$$\rho_e u_e A_e = \rho_t u_t A_t \rightarrow \frac{p_e}{\rho_e u_e} - \frac{p_a}{\rho_a u_a} = \frac{A_t}{A_e} = \frac{M_e}{M_t} \left(1 + \frac{\gamma - 1}{2}M_e^2\right)^{-\frac{\gamma + 1}{2(\gamma - 1)}} \rightarrow M_t = 0.576.$$ 

The value of $\dot{m}$ expressed as $\dot{m}/(\sqrt{\gamma p_c A_e})$ is

$$\dot{m}/(\sqrt{\gamma p_c A_e}) = M_e \left(1 + \frac{\gamma - 1}{2}M_e^2\right)^{-\frac{\gamma + 1}{2(\gamma - 1)}} \simeq 0.119.$$ 

3. Since the value $p_a/p_c = 0.1$ is in the range $p_{ss}/p_c > p_a/p_c > p_{ss1}/p_c$ an oblique shock appears at the exit. The conditions inside the nozzle are simply $M_e = 2.94$ and $M_t = 1$, so that

$$\dot{m}/(\sqrt{\gamma p_c A_e}) = M_e \left(1 + \frac{\gamma - 1}{2}M_e^2\right)^{-\frac{\gamma + 1}{2(\gamma - 1)}} = 0.1447.$$ 

\textsuperscript{1}The value of $M_e = 0.12$ can be alternatively found directly from the tables of steady isentropic flow with $p/p_0 = p_a/p_c = 0.99$. Also, since $A^*$ is conserved, we may write $A^*/A_e = (A^*/A_e)(A_e/A_t) = 2.0558 \times 4 = 8.22$, where $A^*/A_e$ is the value corresponding to $M_e = 0.12$ and $p/p_0 = 0.99$. Using $A^*/A_t = 0.822$ in the tables of steady isentropic flow gives $M_t \simeq 0.576$. 
The shock pressure jump is \( p_a/p_e = (p_a/p_c)/(p_e/p_c) = 0.1/0.0298 = 3.356 \). Using this pressure ratio and \( M_e = 2.94 \) together with the oblique-shock diagram of pressure jump vs incident Mach number yields \( \delta = 22^\circ \) and \( M_2 \simeq 1.9 \).

4. Since \( p_a/p_c < p_{SJ}/p_c \) we find an isentropic expansion wave at the nozzle exit. The conditions inside the nozzle are simply \( M_e = 2.94 \) and \( M_t = 1 \), so that

\[
m/(\sqrt{\gamma p_e p_c A_e}) = M_e \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{-\frac{\gamma + 1}{2(\gamma - 1)}} = 0.1447.
\]

Conservation of stagnation pressure across the expansion provides

\[
p_0 = p_c \rightarrow \frac{p_c}{p_a} = \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right)^{-\frac{\gamma}{\gamma - 1}} = 40 \rightarrow M_2 = 3.056.
\]

Near the edge, the expansion is locally planar and the solution reduces to the Prandtl-Meyer expansion, whose deflection is obtained according to \( \delta = \nu(M_2) - \nu(M_e) \simeq 1.93^\circ \).