Problem 1: Show that the specific impulse $I_{sp} = T / (g\dot{m})$ of a rocket engine with $p_e = p_a$ can be expressed in the form

$$I_{sp} = \left( \frac{2}{\gamma - 1} \right)^{1/2} \frac{\sqrt{\gamma (R_o/W) T_0}}{g} \left[ \left( \frac{p_0}{p_a} \right)^{-\frac{\gamma - 1}{\gamma}} - 1 \right]^{1/2} \left( \frac{p_0}{p_a} \right)^{-\frac{\gamma - 1}{2\gamma}},$$

where $T_0$ and $p_0$ and the stagnation temperature and stagnation pressure in the combustion chamber, $\gamma$ and $W$ are the specific-heat ratio and the mean molecular mass of the exhaust gas, and $R_o = 8.314 \text{ J/(mol K)}$ is the universal gas constant. Use the above expression to determine the specific impulse at take-off of a hydrogen-oxygen rocket engine with $T_0 = 3000 \text{ K}$ and $p_0 = 50 \text{ atm}$. In the evaluation, use $W = 18 \times 10^{-3} \text{ kg/mol}$ and $\gamma = 1.2$ for the exhaust gas, as corresponds to water vapor at high temperature.

Problem 2: Consider a two-stage rocket designed for a 1,000-kg payload. The two stages, which are identical, have $I_{sp} = 400 \text{ s}$, $m_{ST1} = m_{ST2} = 1,000 \text{ kg}$, and $m_{PR1} = m_{PR2} = 5,000 \text{ kg}$.

- Obtain the maximum velocity increment neglecting aerodynamic drag and gravity effects.

Consider now gravity effects.

- To maximize the velocity gain we need to minimize the burning time. Obtain the minimum burning times $t_{b1}$ and $t_{b2}$ if the payload cannot withstand an acceleration greater than $5g$. Assume that the burning rate for each stage is constant, so that $\dot{m}_1 = m_{PR1}/t_{b1}$ and $\dot{m}_2 = m_{PR2}/t_{b2}$.

- Obtain the maximum velocity increment.
**Problem 3:** A rocket must be designed to reach a velocity increment $\Delta u = 5,000$ m/s. The total initial mass of the rocket is $m_o = m_{PL} + m_{ST} + m_{PR} = 10,000$ kg and the structural mass is $m_{ST} = 1,000$ kg. Determine the maximum payload $m_{PL}$ in the following scenarios, using in all cases $\Delta u = 5,000$ m/s, $m_o = 10,000$ kg, $m_{ST} = 1,000$ kg, and $I_{sp} = 450$ s.

We begin by neglecting drag and gravity effects.

1. For a single-stage rocket, show that the propellant mass is given by
   \[ \frac{\Delta u}{g I_{sp}} = \ln \left( \frac{m_o}{m_o - m_{PR}} \right) \rightarrow m_{PR} = m_o \left[ 1 - \exp \left( -\frac{\Delta u}{g I_{sp}} \right) \right]. \]

2. Obtain the value of $m_{PR}$ as well as the payload $m_{PL}$.

Consider next a rocket with two identical stages $m_{ST1} = m_{ST2} = m_{ST}/2 = 500$ kg and $m_{PR1} = m_{PR2} = m_{PR}/2$.

3. By adding the velocity increments $\Delta_1 u$ and $\Delta_2 u$ associated with each one of the two stages show that
   \[ \frac{\Delta u}{g I_{sp}} = \ln \left( \frac{m_o(m_o - m_{ST}/2 - m_{PR}/2)}{(m_o - m_{PR}/2)(m_o - m_{ST}/2 - m_{PR})} \right). \]

4. Solve the above equation to determine the value of $m_{PR}$.

5. Compute the payload $m_{PL}$ as well as the payload ratios $\lambda_1$ and $\lambda_2$ for each rocket stage.

6. Calculate the velocity increments $\Delta_1 u$ and $\Delta_2 u$ associated with each one of the two stages, verifying that $\Delta_1 u + \Delta_2 u = 5,000$ m/s.

Consider now gravity effects for a single-stage rocket.

7. Assuming a constant burning rate $\dot{m}$, show that the burnout time $t_b$ must satisfy
   \[ \frac{t_b}{I_{sp}} \geq \frac{m_{PR}}{N(m_o - m_{PR})} \]
   if the payload cannot withstand an acceleration greater than $N g$.

8. For the minimum possible value of $t_b$, show that the propellant mass is given by
   \[ \frac{\Delta u}{I_{sp} g} = \ln \left( \frac{m_o}{m_o - m_{PR}} \right) - \frac{m_{PR}}{N(m_o - m_{PR})}. \]
   Solving numerically the above equation yields $m_{PR}/m_o = 0.7855$ for $N = 9$.

9. Determine the payload $m_{PL}$ and the burnout time $t_b$.

10. Compute the initial value of the rocket acceleration
   \[ \frac{du}{dt} = \left( \frac{m_{PR} I_{sp}}{m_o t_b} - 1 \right) g. \]