Consider a rocket with specific impulse \( I_{sp} = 400 \) s designed for a 2,000-kg payload with a total initial mass of 12,000 kg and a total structural mass of 2,000 kg. Neglecting aerodynamic drag and gravity effects, obtain the maximum velocity increment corresponding to

- A single-stage design. \( m_{pl} = 2000 \), \( m_{sr} = 2000 \), \( m_{pr} = 8000 \)

- A two-stage design with both stages being identical (i.e. same structure mass and same propellant mass).

- A two-stage design with both stages having the same structural coefficient and the same payload ratio.

\[ \Delta U = U_{eq} \ln \left( \frac{m_0}{m_b} \right) = 41306 \text{ m/s} \]

\[ \Delta U = U_1 + U_2 = U_{eq} \left[ \ln \left( \frac{12000}{8000} \right) + \ln \left( \frac{7000}{3000} \right) \right] = 4911 \text{ m/s} \]

\[ \lambda_1 = \frac{m_{pl} + m_{sr} + m_{pr}}{m_{st1} + m_{pr1}} \Rightarrow \lambda_2 = \frac{m_{pl}}{m_{st2} + m_{pr2}} \Rightarrow m_2 = \sqrt{m_0 m_{pr2}} = 4899 \text{ kg} \]

\[ E_1 = \frac{m_{st1}}{m_{st1} + m_{m1}} = E_2 = \frac{m_{st2}}{m_{st2} + m_{m2}} \Rightarrow m_{st1} = \frac{m_0 - m_{m2}}{m_{st2}} = 2.45 m_{st2} \]

\[ \Rightarrow m_{st1} = 580 \text{ kg} \]

\[ m_{st1} + m_{st2} = 2000 \Rightarrow m_{st1} = 1420 \text{ kg} \]

\[ m_{pr2} = m_{m2} - m_{pl} = 2.319 \text{ kg} \]

\[ m_{pr1} = 8000 - 2.319 = 5681 \text{ kg} \]

\[ \Delta U = U_1 + U_2 = U_{eq} \left[ \ln \left( \frac{12000}{6319} \right) + \ln \left( \frac{4899}{2580} \right) \right] = 2514 + 2514 = 5028 \text{ m/s} \]
We want to build a small solid-propellant rocket for lower-atmosphere operation \((p_o \approx 1 \text{ atm})\) using a nozzle of throat area \(A^* = 23.615 \text{ cm}^2\) and exit area \(A_e = 100 \text{ cm}^2\). If the working fluid is assumed to be a perfect gas with the properties of air at room temperature, determine the following:

- Pressure in the combustion chamber \(p_o\) for which the pressure at the nozzle exit matches the ambient value \(p_e = p_o\)
- Thrust \(T\), for a value of \(T_0 = 36,000\) N, obtain
- Specific impulse \(I_{sp}\)
- Mass flow rate \(m\)

While the temperature in the combustion chamber remains approximately constant during rocket burning, the mass flow rate changes as the solid propellant surface retreats. To investigate the influence of those changes on the performance of the rocket calculate the thrust and the specific impulse if the mass flow rate were either twice or half of the value computed above.

For a choked nozzle with \(A^*/A_e = 0.23615 \Rightarrow \frac{p_e}{p_o} = 0.027224 \Rightarrow \frac{p_e}{p_o} \Rightarrow p_e = 36.73 \text{ atm}\)

\[ T = A^* p_e \left\{ \frac{2x^2}{n-1} \left( \frac{2}{n+1} \right) \left[ 1 - \left( \frac{p_e}{p_o} \right)^{\frac{n-1}{n}} \right] \right\}^{\frac{1}{2}} = 12,763 \text{ KN} \]

\[ A_o = \sqrt{A^*} = 1098 \text{ m/s} \]

\[ \dot{m} = \frac{A^* p_e}{A_o} \left( \frac{2}{n+1} \right)^{\frac{n+1}{2(n-1)}} = 6.48 \text{ kg/s} \]

\[ I_{sp} = \frac{U_e}{g} = \frac{T/\dot{m}}{g} = 200.7 \text{ s} \]

With \(T_0\) constant, \(P_o \propto \dot{m}\) but \(P_e/P_o\) remains constant as long as the nozzle is choked. When we double \(\dot{m}\) \(\Rightarrow P_o = 73.46 \text{ atm}\), \(P_e = 2 \text{ atm}\) under expanded flow.

\[ T' = A^* p_e \left\{ \frac{2x^2}{n-1} \left( \frac{2}{n+1} \right) \left[ 1 - \left( \frac{p_e}{p_o} \right)^{\frac{n-1}{n}} \right] \right\}^{\frac{1}{2}} \]

or \(\dot{T} = \frac{P_o}{p_e} T_i + (p_e - p_o) A_e = 26,54 \text{ KN}\), \(I_{sp} = \frac{T/\dot{m}}{g} = 208.7 \text{ s}\)

When we halve \(\dot{m}\) \(\Rightarrow P_o = 18.365 \text{ atm}\), \(P_e = 0.5 \text{ atm}\) over expanded flow.

\[ T = 5,875 \text{ KN}, \quad I_{sp} = 184.84 \text{ s} \]
Consider a two-stage rocket designed for a 1,000-kg payload. The two stages, which are identical, have $I_{sp} = 400$ s, $m_{st1} = m_{st2} = 1,000$ kg, and $m_{rh1} = m_{rh2} = 5,000$ kg.

- Obtain the maximum velocity increment neglecting aerodynamic drag and gravity effects.

Consider now gravity effects.

- To maximize the velocity gain we need to minimize the burning time. Obtain the minimum burning times $t_{b1}$ and $t_{b2}$ if the payload cannot withstand an acceleration greater than $5g$. Assume that the burning rate for each stage is constant, so that $\dot{m}_1 = m_{rh1}/t_{b1}$ and $\dot{m}_2 = m_{rh2}/t_{b2}$.

- Obtain the maximum velocity increment.

\[
me \frac{dv}{dt} = gI_{sp} \dot{m} = gI_{sp} \frac{dm_m}{dt} \Rightarrow \Delta U = gI_{sp} \ln \left( \frac{m_0}{m_b} \right) \Rightarrow \text{THIS APPLIES FOR EACH STAGE}
\]

\[
\Rightarrow \quad v_{max} = gI_{sp} \left[ \ln \left( \frac{13,000}{8,000} \right) + \ln \left( \frac{7,000}{2,000} \right) \right] = 9.81 \times 400 \left[ \ln \left( \frac{13}{8} \right) + \ln \left( \frac{7}{2} \right) \right] = 6.821 \text{ m/s}
\]

\[
m_0 \frac{dv}{dt} = gI_{sp} \dot{m} - mg \Rightarrow \frac{dv}{dt} = g \left( \frac{I_{sp} \dot{m}}{m_b} - 1 \right)
\]

WITH $\dot{m}$ = CONSTANT THE MAXIMUM ACCELERATION OCCURS WHEN $m_r$ IS MINIMUM (i.e., AT BURNOUT FOR EACH ONE OF THE TWO STAGES)

FIRST STAGE \quad $g \left( \frac{I_{sp} \dot{m}_1}{m_{b1}} - 1 \right) \leq 5g$ \quad WITH $\dot{m}_1 = \frac{m_{rh1}}{t_{b1}}$ \quad $\Rightarrow$ \quad $t_{b1} \geq \frac{I_{sp}}{g} \frac{m_{rh1}}{m_{b1}} =$ $\frac{400}{6} \frac{5,000}{8,000} = 11.66$ s

SECOND STAGE \quad $t_{b2} \geq \frac{I_{sp}}{g} \frac{m_{rh2}}{m_{b2}} = \frac{400}{6} \frac{5,000}{2,000} = 166.7$ sec

\[
\frac{dv}{dt} = -gI_{sp} \frac{dm_m}{dt} - g \Rightarrow \Delta U_1 = g \left[ I_{sp} \ln \left( \frac{m_{b1}}{m_{b1}} \right) - t_{b1} \right] = 1496.4 \text{ m/s}
\]

\[
\Delta U_2 = g \left[ I_{sp} \ln \left( \frac{m_{b2}}{m_{b2}} \right) - t_{b2} \right] = 3280.5 \text{ m/s}
\]

\[
U_{max} = \Delta U_1 + \Delta U_2 = 4777 \text{ m/s}
\]