1 Introduction

Unlike air-breathing engines, to be treated separately, rocket engines carry all of the needed propellant. They generate thrust by imparting momentum to the propellant before it is expelled from the engine. They are fundamental for space flight and also have many applications in atmospheric propulsion, including many of military nature. They are the oldest propulsion technology. The short presentation given below focuses on chemical rockets, which are schematically shown in Fig. 1.

![Figure 1: Schematic view of a liquid-propellant rocket engine (left-hand-side plot) and of a solid-propellant rocket engine (right-hand-side plot).](image)

2 Jet propulsion

All rockets employ a convergent-divergent nozzle to accelerate the exhaust stream in order to achieve supersonic velocities \( u_e > a_e \) at the exit section, as indicated in the schematic view of Fig. 2, representing a rocket engine on a test bench. Significant gas motion occurs only at the exit section, where the gas pressure is \( p_e \), whereas the air surrounding the rocket remains nearly stagnant and with nearly uniform pressure \( p_a \). Flow properties will be assumed to be uniform across the nozzle exit, of cross-sectional area \( A_e \), giving a mass flow rate \( \dot{m} = \rho_e u_e A_e \) in terms of the exit density \( \rho_e \).

A control-volume analysis can be used to compute the thrust generated \( T \). This force is transmitted to the ground through the structure supporting the rocket (in steady flight, the thrust balances the drag force acting on the vehicle). Neglecting viscous and buoyancy forces, an excellent approximation for the large values of the Reynolds and Froude numbers characterizing the flow, reduces the horizontal component \( (x) \) of the momentum equation to the familiar form

\[
\int_{S_e} \rho v_x v \cdot n dS = - \int_{S_e} (p - p_a) n_x dS + T, \tag{1}
\]

where \( n \) is the unit vector normal to the control-volume surface \( S_e \), as indicated in the figure. Straightforward integration yields the expression

\[
T = \dot{m} u_e + (p_e - p_a) A_e. \tag{2}
\]
The first term, $\dot{m}u_e$, represents the momentum flux of the exhaust jet, typically the dominant contribution to the thrust. For a given burning rate $\dot{m}$, there is clear interest in maximizing the exhaust velocity, which explains the need for a convergent-divergent nozzle. In addition, there is a correction in rockets with exit pressure $p_e$ different from the ambient pressure $p_a$. This correction, typically small, is positive in underexpanded nozzles with $p_e > p_a$ and negative in overexpanding nozzles with $p_e < p_a$. Maximum thrust is achieved for expansion conditions such that $p_e = p_a$. One can see this, for example, by noting that thrust is a result of the pressure forces acting on the rocket casing, as shown schematically in the lower plot of Fig. 2. The pressure decreases as the flow expands along the divergent section. In a nozzle with optimum expansion $p_e = p_a$, the pressure everywhere inside the nozzle is higher than the ambient value. Since the force acting on any part of the divergent section yields a positive contribution to the thrust, shortening the nozzle results in a smaller value of $T$. Similarly, if we lengthen the nozzle, the added wall would have an inner pressure smaller than the ambient pressure, resulting in a force opposite to the thrust (to the right, if the rocket is pointing to the left, as in the figure), so that the enlarged nozzle would also provide less thrust.

### 3 Specific impulse

The contribution to the thrust in (2) coming from the pressure mismatch, typically small, can be accounted for by defining an equivalent exhaust velocity

$$u_{eq} = u_e + \frac{(p_e - p_a)A_e}{\dot{m}},$$

such that

$$T = \dot{m}u_{eq}.$$

When the thrust is acting on a rocket, the associated gain of momentum over a period of time, or impulse, is given by

\[ I = \int T \, dt = \int \dot{m} u_{eq} \, dt. \]  \hspace{1cm} (5)

Assuming that \( u_{eq} \) remains constant, it follows that \( I = u_{eq} \int \dot{m} \, dt = m_{PR} u_{eq} \), where \( m_{PR} \) is the mass of propellant burnt. Correspondingly, the impulse per unit mass of propellant is

\[ \frac{I}{m_{PR}} = \frac{T}{\dot{m}} = u_{eq}. \]  \hspace{1cm} (6)

This quantity is often expressed by replacing the mass of propellant by its corresponding weight \( gm_{PR} \), leading to the so-called specific impulse

\[ I_{sp} = \frac{I}{gm_{PR}} = \frac{T}{g \dot{m}} = \frac{u_{eq}}{g}, \]  \hspace{1cm} (7)

which is customarily expressed in seconds.

This specific impulse is a measure of the efficiency of the rocket engine. Since \( u_{eq} \approx u_{e} \), it is clear that one should increase the exit velocity in order to increase the specific impulse. The stagnation enthalpy \( h_0 \) is conserved across the nozzle, where the flow is steady and adiabatic, so that the velocity \( u_{e} \) and enthalpy \( h_e \) at the exit section must satisfy \( h_e + u_e^2/2 = h_0 \), which can be rewritten in the form

\[ u_e = \left( \frac{2h_0}{1 + \frac{2}{\gamma - 1} M_e^2} \right)^{1/2} \]  \hspace{1cm} (8)

in terms of the exit Mach number \( M_e \). Clearly, to maximize \( u_e \) for a given value of \( h_0 \) one should design the exhaust to provide the highest possible value of \( M_e \) by use of convergent-divergent nozzles with large expansion ratios. The maximum possible value of \( u_e \), reached for \( M_e \gg 1 \), is \( u_e = \sqrt{2h_0} \).

For chemical rockets, an estimated value for \( h_0 \) follows from simple energy arguments. In the combustion chamber, where the Mach number is very small, the stagnation enthalpy is approximately equal to the thermal enthalpy of the hot combustion products \( h_0 = h_{init} + Q_R \), involving the initial thermal enthalpy of the propellants prior to combustion \( h_{init} \) and the heat released in the combustion process per unit mass of propellant \( Q_R \), with \( h_{init} \ll Q_R \) in all systems of interest. The value of \( Q_R \) depends on the propellant. For example, \( Q_R \approx 9,000 \) kJ/kg for hydrocarbon-oxygen rockets and \( Q_R \approx 13,400 \) kJ/kg for hydrogen-oxygen rockets, as follows from a simple energy balance involving the enthalpy of formation of the reactants and products. These numbers can be used in \( u_e \leq \sqrt{2h_0} \approx \sqrt{2Q_R} \) to give the estimated maximum values \( u_e \lesssim 4200 \) m/s for hydrocarbon-oxygen systems and \( u_e \lesssim 5100 \) m/s for hydrogen-oxygen systems, corresponding to values of the specific impulse \( I_{sp} \lesssim 420 \) s \( I_{sp} \lesssim 510 \) s, respectively, as follows from (7) with \( u_{eq} \approx u_{e} \). \footnote{These are theoretical upper bounds. Typical values achieved in real systems are \( I_{sp} \lesssim 350 \) s and \( I_{sp} \lesssim 450 \) s for hydrocarbon-oxygen systems and hydrogen-oxygen systems, respectively.}

4 Performance of rocket vehicles

To understand the relevance of the specific impulse and the equivalent velocity in connection with the performance of a rocket engine, it is illustrative to consider as a model problem the vertical launch of a rocket from the Earth’s surface (the following analysis applies, for example, to the launch of a sounding rocket). The mission objective is defined as that of taking a given payload to a given altitude.
4.1 Rocket-mass distribution

The initial mass of the rocket \(m_o = m_{ST} + m_{PL} + m_{PR}\) includes the structure \(m_{ST}\) (engine, tankage, supporting structure, crew, etc), the payload \(m_{PL}\), and the propellant \(m_{PR}\). The mass distribution is characterized by different dimensionless parameters, including the mass ratio

\[
R = \frac{m_o}{m_b} > 1,
\]

where \(m_b = m_{ST} + m_{PL}\) is the rocket mass at the end of the burning stage, the structural coefficient

\[
\varepsilon = \frac{m_{ST}}{m_{ST} + m_{PR}},
\]

and the payload ratio

\[
\lambda = \frac{m_{PL}}{m_{ST} + m_{PR}},
\]

with

\[
R = \frac{1 + \lambda}{\varepsilon + \lambda},
\]

as follows from the above definitions.

4.2 Problem formulation and useful simplifications

Newton’s second law applied to the vertical motion of the rocket takes the form

\[
m_R \frac{du_R}{dt} = T - m_R g - D,
\]

where \(m_R(t)\) and \(u_R(t)\) denote the instantaneous values of the rocket mass and vertical rocket speed, respectively. The thrust can be expressed using (6) as the product \(T = \dot{m}u_{eq}\) of the propellant burning rate \(\dot{m}\) and the equivalent exhaust velocity \(u_{eq}\), both assumed to be constant. The drag force acting on the rocket, given by \(D = C_D \frac{1}{2} \rho u_R^2 A_f\) in terms of the drag coefficient \(C_D\) and frontal cross-sectional area \(A_f\), changes in time as the ambient density \(\rho\) decreases and the rocket speed \(u_R\) increases. The gravitational acceleration \(g\) also changes from its value at the earth’s surface \(g_E \simeq 9.81\) m/s\(^2\) according to

\[
g = \left(\frac{R_E}{R_E + h}\right)^2 g_E,
\]

where \(R_E \simeq 6,400\) km is the radius of the earth and \(h\) is the rocket altitude. These changes are clearly small when \(h \ll R_E\) (e.g. \(g \simeq 0.95g_E\) for \(h = 100\) miles), and can be correspondingly neglected in the following description\(^2\), where the approximation \(g = g_E\) is adopted. To facilitate the analysis, drag forces

\(^2\)Accounting for changes of \(g\) is important when the rocket is traveling to distances comparable to \(R_E\), for example, in the calculation of the escape velocity \(u_{esc}\) (the initial firing velocity needed to overcome the earth’s gravitational pull) through integration of

\[
\frac{du_R}{dt} = -\left(\frac{R_E}{R_E + h}\right)^2 g_E, \quad \frac{dh}{dt} = u_R
\]

with initial conditions \(u_R(0) = u_{init} = h(0) = 0\). A first integral provides

\[
\frac{u^2_{init} - u_R^2}{2} = g_E R_E \left(1 - \frac{1}{1 + h/R_E}\right)
\]

for the variation of \(u_R\) with \(h\). Setting \(u_R = 0\) in the above expression provides the initial velocity needed to reach a given altitude. In particular, reaching \(h = \infty\) requires that \(u_{init} > u_{esc} = \sqrt{2g_E R_E} \simeq 11,200\) m/s.
will also be neglected below, reducing the problem to that of integrating

\[ \frac{d u_R}{dt} = m R d u_R = \dot{m} u_{eq} - m R g_E \]  \tag{15} \\
\frac{d h}{dt} = u_R \tag{16} \\
\frac{d m_R}{dt} = - \dot{m} \tag{17}

with initial conditions

\[ u_R(0) = h(0) = m_R(0) - m_o = 0. \]  \tag{18}

### 4.3 Closed-form solution

The problem defined in (15)-(18) admits a closed-form solution. In general, two different stages must be considered. There is an initial burning stage with constant thrust \( T = \dot{m} u_{eq} \) ending at \( t_b = m_{PR}/\dot{m} \) followed by a non-burning stage with \( T = 0 \). During the burning stage \( 0 \leq t \leq t_b \) the mass of the rocket decreases linearly with time according to \( m_R = m_o - \dot{m} t \), as follows from integration of (17). This result can be used in (15) to yield

\[ u_R = - u_{eq} \ln \left( 1 - \frac{\dot{m} t}{m_o} \right) - g_E t \]  \tag{19}

upon integration with the initial condition \( u_R(0) = 0 \). Substitution of (19) into (16) followed by integration with \( h(0) = 0 \) provides

\[ h = u_{eq} m_o \frac{m}{\dot{m}} \ln \left( 1 - \frac{\dot{m} t}{m_o} \right) + u_{eq} t - g_E t^2 / 2 \]  \tag{20}

for the variation of altitude during the first stage. At burnout (i.e. \( t = t_b \)) the resulting rocket velocity and altitude are

\[ u_R = u_b = u_{eq} \ln(R) - g_E t_b \]  \tag{21}

and

\[ h = h_b = u_{eq} t_b \left( 1 - \ln(R) \frac{R}{R - 1} \right) - g_E t_b^2 / 2 \]  \tag{22}

in terms of the mass ratio \( R = m_o/m_b \).

Before proceeding to analyze the rocket evolution for \( t > t_b \), it is of interest to look in detail at the velocity achieved at burnout to get a feel for the technical challenges we are facing. To that end, (21) can be used to write

\[ R = \exp \left( \frac{u_b}{u_{eq}} + \frac{t_b}{I_{sp}} \right) > \exp \left( \frac{u_b}{u_{eq}} \right) \]  \tag{23}

as an estimated lower bound for the mass ratio needed to achieve a given \( u_b \). Since \( R = (1 - m_{PR}/m_o)^{-1} \), it follows that

\[ \frac{m_{PR}}{m_o} > 1 - \exp \left( - \frac{u_b}{u_{eq}} \right). \]  \tag{24}

Consider now as an illustrative example the use of hydrocarbon-oxygen liquid-propellant rocket engines (\( u_{eq} \approx 3500 \text{ m/s} \)) for interplanetary travel (\( u_b = u_{esc} \approx 11,200 \text{ m/s} \)). The above expression indicates that \( m_{PR}/m_o > 0.96 \), that is, the propellant amounts to 96% of the initial weight of the rocket, the remaining 4% being distributed between the rocket structure and the payload.
During the post-burning stage $t > t_b$ we find $\dot{m} = 0$ and $m_R = m_b$. Eliminating the time between (15) and (16) and integrating gives the energy conservation law

$$\frac{1}{2}u_R^2 + g_E h = \frac{1}{2}u_b^2 + g_E h_b,$$

which relates the rocket velocity and the rocket altitude. Since we are using the approximation $g = g_E$, the validity of the above expression is restricted to cases with $h \ll R_E$, thereby excluding interplanetary travel, for example.

### 4.4 Maximum altitude

According to (25), the maximum altitude, reached when $u_R = 0$, is given by

$$h_{max} = h_b + \frac{u_b^2}{2g_E} = \frac{u_{eq}^2}{2g_E} \ln^2(R) - u_{eq} t_b \left[ \frac{R}{R - 1} \ln(R) - 1 \right].$$

The term in square brackets in (26) can be shown to be positive for all $R > 1$. Therefore, it is clear that to maximize $h_{max}$ for a given $u_{eq}$ one should minimize the burning time $t_b$. Ideally, one would like to consume all of the propellant instantaneously. However, that does not seem to be a good idea, partly because it would imply an infinite propellant flow rate $\dot{m} = m_{PR}/t_b$ and partly because it would involve an infinite thrust $T = m_{PR} u_{eq}/t_b$ and, therefore, an infinite acceleration, compromising the rocket integrity. In many cases, the minimum burning time $(t_b)_{min}$ is associated with payload acceleration tolerance. For instance, if the payload must not suffer an acceleration greater than $N g_E$, then the minimum burning time is determined by the condition $T/m_R = (m_{PR} u_{eq})/(m_R t_b) < N g_E$. Since the acceleration is maximum at burnout, when $m_R = m_b$, it follows that

$$(t_b)_{min} = \frac{(R - 1) u_{eq}}{N g_E} = (R - 1) I_{sp} / N,$$

so that the maximum altitude that can be attained for that payload tolerance is

$$h_{max} = g_E I_{sp}^2 \left[ \frac{\ln^2(R)}{2} - \frac{R [\ln(R) - 1]}{N} + 1 \right].$$

### 4.5 Multistage rockets

As indicated above below (24), in missions involving chemical rockets most of the initial mass of the rocket is propellant. The tanks and support structure for the propellant also constitute a significant part of the weight, often larger than the payload itself. Multistage rockets are designed to address this problem by discarding portions of the structure as they become empty, so that thrust is not wasted in accelerating this dead mass once it is no longer useful. A multistage rocket comprises a number of individual stages, each one with its own structure, tank, and engine. They are numbered in the order they are fired.

To illustrate the gains associated with multistaging we can consider a rocket with equivalent velocity $u_{eq}$ with initial mass distribution $m_o = m_{PL} + m_{ST} + m_{PR}$. For simplicity, the weight of the rocket and the drag force will be discarded in analyzing the rocket acceleration. A combination of (15) and (17) leads to

$$m_R \frac{d u_R}{dt} = -u_{eq} \frac{d m_R}{dt},$$

with can be integrated with initial conditions $u_R(0) = m_R(0) = m_o = 0$ to give

$$u_R = u_{eq} \ln \left( \frac{m_o}{m_R} \right),$$
finally yielding
\[ u_b = u_{eq} \ln \left( \frac{m_o}{m_o - m_{PR}} \right) = u_{eq} \ln R = u_{eq} \ln \left( \frac{1 + \lambda}{\varepsilon + \lambda} \right) \] (31)

at the end of the burning period, where \( R \) is the rocket mass ratio, related to the structural coefficient \( \varepsilon \) and the payload ratio \( \lambda \) through (12). For example, evaluating (31) for a rocket with \( m_{PL} = 1,000 \) kg, \( m_{ST} = 3,000 \) kg, and \( m_{PR} = 20,000 \) kg, yields a rocket velocity \( u_b \approx 1.79u_{eq} \) at burnout.

Consider now a two-stage rocket with the same overall mass distribution, such that \( m_{ST} = m_{ST1} + m_{ST2} \) and \( m_{PR} = m_{PR1} + m_{PR2} \). The two stages are supposed to have the same equivalent velocity \( u_{eq} \) as that of the single-stage rocket analyzed above. Acceleration occurs in two stages with corresponding velocity increments \( \Delta_1 u \) and \( \Delta_2 u \), such that the final velocity at burnout is \( u_b = \Delta_1 u + \Delta_2 u \). The analysis of the first stage leads to
\[ \Delta_1 u = u_{eq} \ln \left( \frac{m_o}{m_o - m_{PR1}} \right) \] (32)

while the analysis of the second stage, which begins once the structure of the first stage is discarded, gives
\[ \Delta_2 u = u_{eq} \ln \left( \frac{m_{PL} + m_{ST2} + m_{PR2}}{m_{PL} + m_{ST2}} \right) . \] (33)

For example, if we assume that \( m_{ST1} = m_{ST2} = 1,500 \) kg and \( m_{PR1} = m_{PR2} = 10,000 \) kg, the velocity gains are \( \Delta_1 u \approx 0.54u_{eq} \) and \( \Delta_2 u \approx 1.61u_{eq} \), respectively, thereby yielding a final velocity \( u_b = 2.15u_{eq} \), about 20% larger than that of the single-stage rocket.

In general, for a multistage rocket the velocity increment of stage \( i \) can be expressed in the form
\[ \Delta_i u = u_{eq} \ln \left( \frac{m_{o_i}}{m_{o_i} - m_{PR_i}} \right) = u_{eq} \ln \left( \frac{1 + \lambda_i}{\varepsilon_i + \lambda_i} \right) , \] (34)

where in defining the payload ratio \( \lambda_i \) of stage \( i \) the payload is the mass of all subsequent stages. For example, for a two-stage rocket
\[ \lambda_1 = \frac{m_{PL} + m_{ST2} + m_{PR2}}{m_{ST1} + m_{PR1}} \quad \text{and} \quad \lambda_2 = \frac{m_{PL}}{m_{ST2} + m_{PR2}} \] (35)

while
\[ \varepsilon_1 = \frac{m_{ST1}}{m_{ST1} + m_{PR1}} \quad \text{and} \quad \varepsilon_2 = \frac{m_{ST2}}{m_{ST2} + m_{PR2}} . \] (36)

It can be shown that if the structural coefficients \( \varepsilon_i \) and equivalent velocities \( u_{eq} \) of all stages are equal, then the solution that gives maximum acceleration is that with identical stage payload ratios \( \lambda_i \). For instance, for the two-stage rocket of the previous example, the solution with \( \lambda_1 = \lambda_2 \) and \( \varepsilon_1 = \varepsilon_2 \) corresponds to a mass distribution \( m_{ST1} = 2,491.5 \) kg, \( m_{ST2} = 508.5 \) kg, \( m_{PR1} = 16,609.5 \) kg, and \( m_{PR2} = 3,390.5 \) kg, for which \( \Delta_1 u = \Delta_2 u \approx 1.18u_{eq} \) and \( u_b \approx 2.36u_{eq} \), about 32% larger than that of the single-stage rocket.

5 Performance of chemical rocket engines

5.1 Enthalpy-entropy diagram for the gas evolution

The evolution of the gas in the rocket engine can be conveniently represented in an enthalpy-entropy diagram, shown in Fig. 3. The reactants are introduced in the combustion chamber, where they react with nearly uniform pressure \( p_0 \) because the prevailing Mach number is very small. Conservation of energy
leads to $\dot{m}(h_{02} - h_{01}) = \dot{m}Q_R$, where $h_{01} = c_pT_{01}$ and $h_{02} = c_pT_{02}$ are the stagnation enthalpy of the reactants and products, respectively, with $T_{01}$ and $T_{02}$ being the corresponding stagnation temperatures (approximately equal to the temperature). Since $T_{01} \ll Q_R/c_p$, it follows that $T_{02} \approx Q_R/c_p$. In the following we shall use $T_0 = T_{02}$ to denote the stagnation temperature of the combustion products, which is the stagnation temperature for the flow in the nozzle.

\[ A^* = A_t \quad M_e > 1 \quad p_e \]

\[ h \quad c_pT_0 \]

\[ Q_R \quad u_2^2 \]

\[ s \]

Figure 3: Evolution of the gas inside the rocket engine.

In the $h - s$ diagram of Fig. 3, the combustion process corresponds to an evolution along the curve of constant pressure $p_0$, with the vertical displacement being the heat of reaction $Q_R$. The nozzle is assumed to be isentropic, with the result that the expansion of the combustion products across the nozzle becomes a vertical line, with the change of thermal enthalpy (i.e. the vertical displacement) corresponding to the gain in kinetic energy $u_2^2/2$.

### 5.2 Steady isentropic nozzle flow

The flow in the nozzle is steady and inviscid, so that we can use

\[ \left(\frac{p_0}{p}\right)^{\frac{\gamma - 1}{\gamma}} = \frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2 \quad (37) \]

along with

\[ \dot{m} = \frac{\dot{m}}{Ap_0/\sqrt{R_gT_0/\gamma}} = M \left(1 + \frac{\gamma - 1}{2}M^2\right)^{-\frac{\gamma + 1}{2(\gamma - 1)}} = \left(\frac{2}{\gamma - 1}\right)^{1/2} \left(p/p_0\right)^{1/\gamma} \left[1 - \left(p/p_0\right)^{\frac{\gamma - 1}{\gamma}}\right]^{1/2}, \quad (38) \]
and
\[ A = \frac{1}{M} \left[ \frac{2}{\gamma + 1} \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma + 1}{\gamma - 1}} \right]^{\frac{\gamma + 1}{\gamma - 1}} = \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{\gamma - 1}} \left( \frac{\gamma - 1}{2} \right)^{1/2} \frac{(p/p_0)^{1/\gamma}}{\sqrt{1 - (p/p_0)^{\frac{\gamma - 1}{\gamma}}}} \] (39)

involving the local values of \( A, M, p, \) and \( T \) at any section and the critical area \( A^* \). At the exit section, where the pressure is \( p_e \), the above equations can be used to yield

\[ u_e = M_e \sqrt{\gamma R T_e} = \sqrt{\frac{2 \gamma R T_0}{\gamma - 1}} \left[ 1 - \left( \frac{p_e}{p_0} \right)^{\frac{\gamma - 1}{\gamma}} \right] \] (40)

for the exhaust speed \( u_e = M_e a_e \). As can be seen, since the gas constant \( R_g \) is related to the universal gas constant \( R_o \) through \( R_g = R_o / W \), for a given \( T_0 \) it is convenient to reduce the molecular mass of the propellant \( W \) to increase the velocity (and therefore the thrust). The previous estimate \( T_0 = Q R / c_p \) can be used in (40) to express the exit velocity in the alternative form

\[ u_e = \sqrt{2 Q R} \left[ 1 - \left( \frac{p_e}{p_0} \right)^{\frac{\gamma - 1}{\gamma}} \right]. \] (41)

Rocket nozzles are always choked. The corresponding propellant burning rate can be obtained by evaluating (38) at the throat, where \( M_t = 1 \) and \( A_t = A^* \), to give

\[ \dot{m} = p_0 A^* \left[ \frac{\gamma}{R g T_0} \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{\gamma - 1}} \right]^{1/2}. \] (42)

This last expression can be rearranged to define the so-called characteristic velocity

\[ c^* = \frac{p_0 A^*}{\dot{m}} = \left[ \frac{R g T_0}{\gamma} \left( \frac{\gamma + 1}{2} \right)^{\frac{\gamma + 1}{\gamma - 1}} \right]^{1/2} = \left[ \frac{\gamma - 1}{\gamma^2} \left( \frac{\gamma + 1}{2} \right)^{\frac{\gamma + 1}{\gamma - 1}} Q_R \right]^{1/2}, \] (43)

which serves to measure the combustion-chamber performance.

### 5.3 The thrust coefficient

For a rocket engine with given combustion-chamber pressure \( p_0 \) and given throat area \( A^* \) it is convenient to express the thrust in dimensionless form with use of the thrust coefficient

\[ \frac{T}{p_0 A^*} = \left\{ \frac{2 \gamma^2}{\gamma - 1} \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{\gamma - 1}} \left[ 1 - \left( \frac{p_e}{p_0} \right)^{\frac{\gamma - 1}{\gamma}} \right] \right\}^{1/2} + \frac{A_e}{A^*} \left( \frac{p_e}{p_0} \right)^{1/\gamma}, \] (44)

obtained by substituting (40) and (42) into (2). Since the area ratio \( A_e/A^* \) is related to \( p_e/p_0 \) by

\[ A_e/A^* = \left( \frac{2}{\gamma + 1} \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \left( \frac{\gamma - 1}{2} \right)^{1/2} \frac{(p_e/p_0)^{-1/\gamma}}{\sqrt{1 - (p_e/p_0)^{\frac{\gamma - 1}{\gamma}}}}, \] (45)
determined by evaluating (39) at the exit section, we can finally write

\[
\frac{T}{p_0 A^*} = \left\{ \frac{2\gamma^2}{\gamma - 1} \left( \frac{2}{\gamma + 1} \right)^{\gamma+1} \left[ 1 - \left( \frac{p_e}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{1/2} + \left( \frac{2}{\gamma + 1} \right)^{\gamma+1} \left( \frac{\gamma - 1}{2} \right)^{1/2} \left( \frac{p_e}{p_0} - \frac{p_a}{p_0} \right) \frac{(p_e/p_0)^{-1/\gamma}}{\sqrt{1 - (p_e/p_0)^{\frac{\gamma-1}{\gamma}}}}
\]

(46)
as an expression for the thrust coefficient. The above expressions (40), (42) and (44) indicate that changes in the temperature of the combustion products \(T_0\) modify the values of the exit velocity \(u_e\) and of the propellant mass flow rate \(\dot{m}\), but not the resulting thrust, which is independent of \(T_0\), as seen in (44).

Equation (46) gives the variation of \(T/(p_0 A^*)\) with \(p_e/p_0\). As can be shown by solving

\[
\frac{d[T/(p_0 A^*)]}{d(p_e/p_0)} = 0
\]

for fixed values of \(\gamma\) and \(p_a/p_0\), the maximum possible thrust

\[
\left( \frac{T}{p_0 A^*} \right)_{max} = \left\{ \frac{2\gamma^2}{\gamma - 1} \left( \frac{2}{\gamma + 1} \right)^{\gamma+1} \left[ 1 - \left( \frac{p_a}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{1/2}
\]

(47)
corresponds to conditions such that \(p_e = p_a\). Correspondingly, the nozzle area ratio that maximizes thrust is given by

\[
\left( \frac{A_e}{A^*} \right)_{T_{max}} = \left( \frac{2}{\gamma + 1} \right)^{\gamma+1} \left( \frac{\gamma - 1}{2} \right)^{1/2} \frac{(p_a/p_0)^{-1/\gamma}}{\sqrt{1 - (p_a/p_0)^{\frac{\gamma-1}{\gamma}}}},
\]

(48)
as follows from (45).

5.4 Malina’s diagram

For a given combustion-chamber pressure \(p_0\) and a given ambient pressure \(p_a\) it is of interest to investigate the variation of the thrust with the nozzle area ratio \(A_e/A^*\). We begin by considering a convergent nozzle, i.e. \(A_e = A^*\) and \(p_e/p_0 = [2/(\gamma + 1)]^{\gamma/(\gamma-1)}\), for which the thrust coefficient (46) reduces to

\[
\left( \frac{T}{p_0 A^*} \right)_{conv} = 2 \left( \frac{2}{\gamma + 1} \right)^{\gamma+1} - \frac{p_a}{p_0}
\]

(49)

Dividing (44) by (49) gives

\[
\frac{T}{T_{conv}} = \left\{ \frac{2\gamma^2}{\gamma - 1} \left( \frac{2}{\gamma + 1} \right)^{\gamma+1} \left[ 1 - \left( \frac{p_a}{p_0} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{1/2} + \left( \frac{p_a}{p_0} - \frac{p_a}{p_0} \right) A_e A^*
\]

\[
2 \left( \frac{2}{\gamma + 1} \right)^{\gamma+1} - \frac{p_a}{p_0}
\]

(50)
The above expression, supplemented with (45) for the evaluation of \(p_e/p_0\) as a function of \(A_e/A^*\), enables the computation of the thrust of a convergent-divergent nozzle relative to that of a convergent nozzle with the same throat area (for a given value of \(p_a/p_0\)). The resulting diagram, due to Frank Malina, is presented in Fig. 4.
Besides solid curves corresponding to different values of $p_a/p_0$, the plot includes an upper dashed curve representing the maxima of the different curves. As indicated in the upper panels $a - d$, as $A_e/A^*$ increases for each value of $p_a/p_0$, the solution at the exit evolves from an under-expanded jet ($a$), to a supersonic jet with $p_a = p_e$ ($b$), to an over-expanded jet with an oblique shock of increasing incident angle ($c$), eventually reaching conditions for which the shock becomes normal ($d$). Since there is no interest in nozzles with internal shock waves, the solid curves are not plotted beyond these limiting conditions, indicated by a second dashed curve. The maximum of $A_e/A^*$ for each value of $p_a/p_0$, at which a normal shock emerges at the exit, can be evaluated from (39) written in the form

$$\frac{A_e}{A^*} = \frac{1}{M_e} \left[ \frac{2}{\gamma + 1} \left( 1 + \frac{\gamma - 1}{2} M_e^2 \right) \right]^{\frac{\gamma + 1}{2(\gamma - 1)}} \tag{51}$$

with the exit Mach number $M_e$ computed from

$$\frac{p_a}{p_0} = \frac{p_a}{p_e} = \frac{(2\gamma M_e^2 + 1 - \gamma)/(\gamma + 1)}{(1 + \frac{\gamma - 1}{2} M_e^2)^{\frac{1}{\gamma - 1}}} \tag{52}$$
Figure 4: The normalized thrust of a convergent-divergent nozzle as a function of $A_e/A^*$ for different values of $p_a/p_0$ in the range $0.001 \leq p_a/p_0 \leq 0.05$ (Frank Malina, Journal of the Franklin Institute 230:433-454, 1940).