In cylindrical coordinates, $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$. Use the expressions

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)$$

and

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

to derive the three components of the momentum equation

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}.$$

In the derivation, note that

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \qquad \text{and} \qquad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r.$$

$$\bar{v}\cdot\nabla = V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta} + V_z\frac{\partial}{\partial z}$$

$$(\bar{v}\cdot\nabla)(V_r\bar{e}_r) = \left(V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z}\right)\bar{e}_r + \frac{V_\theta V_r}{r}\bar{e}_\theta$$

$$(\bar{v}\cdot\nabla)(V_\theta\bar{e}_\theta) = \left(V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z}\right)\bar{e}_\theta - \frac{V_\theta^2}{r}\bar{e}_r$$

$$(\bar{v}\cdot\nabla)(V_z\bar{e}_z) = \left(V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right)\bar{e}_z$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\bar{v} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_r}{\partial r}\right)\bar{e}_r + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right)\bar{e}_\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)\bar{e}_z$$

$$\frac{\partial}{\partial \theta}\left(V_r\bar{e}_r + V_\theta\bar{e}_\theta + V_z\bar{e}_z\right) = \left(\frac{\partial V_r}{\partial \theta} - V_\theta\right)\bar{e}_r + \left(\frac{\partial V_\theta}{\partial \theta} + V_r\right)\bar{e}_\theta + \frac{\partial V_z}{\partial \theta}\bar{e}_z$$

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\bar{v}) = \frac{1}{r^2}\left(\frac{\partial^2 V_r}{\partial \theta^2} - 2\frac{\partial V_\theta}{\partial \theta} - V_r\right)\bar{e}_r + \frac{1}{r^2}\left(\frac{\partial^2 V_\theta}{\partial \theta^2} + 2\frac{\partial V_r}{\partial \theta} - V_\theta\right)\bar{e}_\theta + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2}\bar{e}_z$$

$$\frac{\partial^2}{\partial z^2}\bar{v} = \frac{\partial^2 V_r}{\partial z^2}\bar{e}_r + \frac{\partial^2 V_\theta}{\partial z^2}\bar{e}_\theta + \frac{\partial^2 V_z}{\partial z^2}\bar{e}_z$$

$$r: \quad \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_r}{\partial r}\right) - \frac{V_r}{r^2}}_{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right)} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial^2 V_r}{\partial z^2}\right]$$

$$\theta: \quad \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_\theta V_r}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial P}{\partial \theta} + \nu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right) - \frac{V_\theta}{r^2}}_{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right)} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2}\right]$$

$$z: \quad \frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 V_z$$

For axisymmetric flow $\mathbf{v} = v_r(r,z)\mathbf{e}_r + v_z(r,z)\mathbf{e}_z$ show that

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{v} = \omega_\theta \mathbf{e}_\theta = \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta.$$

If the flow is inviscid, show that the vorticity equation reduces to

$$\frac{D}{Dt}\left(\frac{\omega_\theta}{r}\right) = \frac{\partial}{\partial t}\left(\frac{\omega_\theta}{r}\right) + v_r\frac{\partial}{\partial r}\left(\frac{\omega_\theta}{r}\right) + v_z\frac{\partial}{\partial z}\left(\frac{\omega_\theta}{r}\right) = 0.$$

$$\omega = \frac{1}{r}\begin{vmatrix} \bar{e}_r & r\bar{e}_\theta & \bar{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r(r,z) & \cancel{r V_\theta}^{\,0} & V_z(r,z) \end{vmatrix} = \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\bar{e}_\theta$$

$$\frac{D\bar{\omega}}{Dt} = (\bar{\omega}\cdot\nabla)\bar{V} \Rightarrow \frac{\partial \bar{\omega}}{\partial t} + (\bar{V}\cdot\nabla)\bar{\omega} = (\bar{\omega}\cdot\nabla)\bar{V}$$

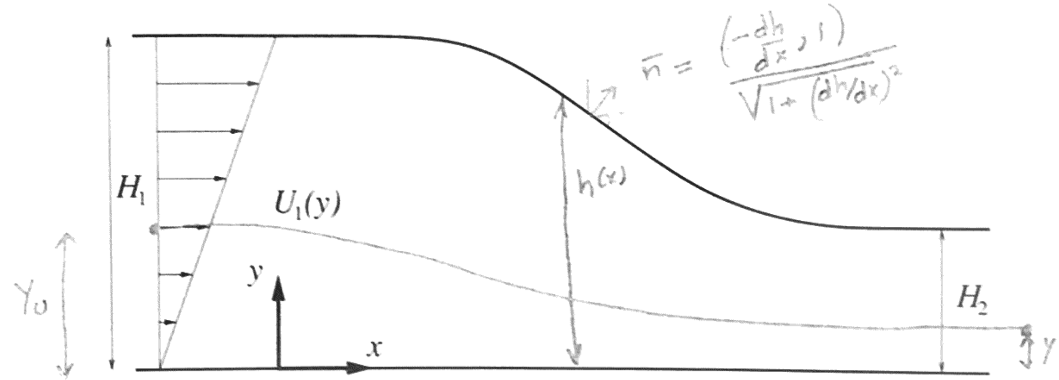
$$\bar{V}\cdot\nabla = V_r\frac{\partial}{\partial r} + V_z\frac{\partial}{\partial z} \Rightarrow (\bar{V}\cdot\nabla)\bar{\omega} = \left(V_r\frac{\partial \omega_\theta}{\partial r} + V_z\frac{\partial \omega_\theta}{\partial z}\right)\bar{e}_\theta$$

$$\bar{\omega}\cdot\nabla = \frac{\omega_\theta}{r}\frac{\partial}{\partial \theta} \Rightarrow (\bar{\omega}\cdot\nabla)\bar{V} = \frac{\omega_\theta}{r}V_r\bar{e}_\theta$$

$$\theta: \quad \frac{\partial \omega_\theta}{\partial t} + V_r\frac{\partial \omega_\theta}{\partial r} + V_z\frac{\partial \omega_\theta}{\partial z} - \frac{\omega_\theta}{r}V_r = 0 \Rightarrow \boxed{\frac{\partial}{\partial t}\left(\frac{\omega_\theta}{r}\right) + V_r\frac{\partial}{\partial r}\left(\frac{\omega_\theta}{r}\right) + V_z\frac{\partial}{\partial z}\left(\frac{\omega_\theta}{r}\right) = 0}$$

DIVIDING BY r

Consider the planar inviscid flow of a constant density fluid through a contraction in a channel. The velocity profile upstream from the contraction (i.e. as $x \to -\infty$ for $0 \le y \le H_1$) is given by $\mathbf{v} = U_1(y)\mathbf{e}_x = Ay\mathbf{e}_x$ and the pressure is $p = p_1$, where A and p_1 are constant. Obtain the velocity profile $\mathbf{v} = U_2(y)\mathbf{e}_x$ and the pressure $p = p_2$ downstream from the contraction (i.e. as $x \to +\infty$ for $0 \le y \le H_2$).



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$x \to -\infty,\ 0<y<H_1:\quad u = U_1(y),\ v = 0$

$x \to +\infty,\ 0<y<H_2:\quad \frac{\partial u}{\partial x} = 0,\ v = 0$

$-\infty < x < \infty:\quad y = 0: v = 0$

$y = h(x): \bar{v}\cdot\bar{n} = 0 \Rightarrow \boxed{u\frac{dh}{dx} = v}$

$$\underbrace{\int_0^{h(x)} \frac{\partial u}{\partial x}\,dy}_{\frac{d}{dx}\int_0^h u\,dy - \frac{dh}{dx}u(x,h)} + \overbrace{v(x,h)}^{u(x,h)\frac{dh}{dx}} - \cancel{v(x,0)}^{\,0} = 0 \Rightarrow \frac{d}{dx}\int_0^h u\,dy = 0 \Rightarrow \boxed{\int_0^h u\,dy = \int_0^{H_1} U_1\,dy}$$

$$\bar{v}\cdot\nabla\left(\frac{u^2+v^2}{2} + \frac{p}{\rho}\right) = 0 \Rightarrow \boxed{\frac{u^2+v^2}{2} + \frac{p}{\rho} = C_e}$$

$$\frac{p_2}{\rho} + \frac{U_2^2(y)}{2} = \frac{p_1}{\rho} + \frac{U_1^2(Y_0)}{2} \Rightarrow \boxed{U_2 = \sqrt{U_1^2(Y_0) + 2\frac{p_1-p_2}{\rho}}}$$

$$U_1\,dY_0 = U_2\,dy \Rightarrow dy = \frac{U_1(Y_0)\,dY_0}{\sqrt{U_1^2(Y_0) + 2\frac{p_1-p_2}{\rho}}} \Rightarrow \boxed{H_2 = \int_0^{H_1}\frac{U_1(Y_0)\,dY_0}{\sqrt{U_1^2(Y_0) + 2\frac{p_1-p_2}{\rho}}}}$$

→ PROVIDES $2\frac{p_1-p_2}{\rho}$

$$\boxed{y = \int_0^{Y_0}\frac{U_1(Y_0)\,dY_0}{\sqrt{U_1^2(Y_0) + 2\frac{p_1-p_2}{\rho}}}}$$ → PROVIDES $Y(Y_0)$

$U_2 = \sqrt{U_1^2(Y_0) + 2\frac{p_1-p_2}{\rho}}$ → GIVES $U_2(Y)$

FOR $U_1 = AY_u \rightarrow Y = \int_0^{Y_u} \frac{U_1(Y_u)\,dY_u}{\sqrt{U_1^2 + 2\frac{P_1-P_2}{\rho}}} \Rightarrow Y = \sqrt{Y_u^2 + \frac{2(P_1-P_2)}{\rho A^2}} - \sqrt{\frac{2(P_1-P_2)}{\rho A^2}}$

$Y = H_2$, $Y_u = H_1$ $\rightarrow$ $\boxed{\frac{2(P_1-P_2)}{\rho A^2} = \left(\frac{H_1^2-H_2^2}{2H_2}\right)^2}$

$\hookrightarrow Y_u^2 = Y\left(Y + \frac{H_1^2-H_2^2}{H_2}\right)$

$$U_2 = \sqrt{U_1^2(Y_u) + 2\frac{P_1-P_2}{\rho}} = \sqrt{A^2 Y_u^2 + A^2\left(\frac{H_1^2-H_2^2}{2H_2}\right)^2} = A\sqrt{\left(Y + \frac{H_1^2-H_2^2}{2H_2}\right)^2} = \boxed{A\left(Y + \frac{H_1^2-H_2^2}{2H_2}\right)}$$

THE SOLUTION CAN ALSO BE OBTAINED (MORE READILY) BY NOTING THAT

$\bar{V}\cdot\nabla\omega = 0 \rightarrow \omega = \omega_0 = -A = \frac{\partial V}{\partial y} - \frac{\partial U}{\partial y}$

AS $y \rightarrow \infty$ $V \rightarrow 0 \rightarrow U = AY + C$

USING CONTINUITY $\int_0^{H_1} AY\,dY = \int_0^{H_2} (AY + C)\,dY \rightarrow \boxed{C = \frac{A}{2}\,\frac{H_1^2-H_2^2}{H_2}}$

USING BERNOULLI ALONG ANY STREAMLINE DETERMINES $\boxed{\frac{P_1-P_2}{\rho} = \frac{C^2}{2} = \frac{A^2}{2}\left(\frac{H_1^2-H_2^2}{2H_2}\right)^2}$

$\left\{ \frac{P}{\rho} + \frac{|\bar{V}|^2}{2} = C_0 \right.$