In cylindrical coordinates, \( \mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \). Use the expressions

\[
\nabla = \left( \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right)
\]

and

\[
\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}
\]

to derive the three components of the momentum equation

\[
(\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}.
\]

In the derivation, note that

\[
\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \quad \text{and} \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r.
\]
For axisymmetric flow $\mathbf{v} = v_r(r, z)\mathbf{e}_r + v_z(r, z)\mathbf{e}_z$ show that

$$\omega = \nabla \wedge \mathbf{v} = \omega_\theta \mathbf{e}_\theta = \left( \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \mathbf{e}_\theta.$$ 

If the flow is inviscid, show that the vorticity equation reduces to

$$\frac{D}{Dt} \left( \frac{\omega_\theta}{r} \right) = \frac{\partial}{\partial t} \left( \frac{\omega_\theta}{r} \right) + v_r \frac{\partial}{\partial r} \left( \frac{\omega_\theta}{r} \right) + v_z \frac{\partial}{\partial z} \left( \frac{\omega_\theta}{r} \right) = 0.$$
Consider the planar inviscid flow of a constant density fluid through a contraction in a channel. The velocity profile upstream from the contraction (i.e., as \( x \to -\infty \) for \( 0 \leq y \leq H_1 \)) is given by \( \mathbf{v} = U_1(y)\mathbf{e}_x = Ay\mathbf{e}_x \) and the pressure is \( p = p_1 \), where \( A \) and \( p_1 \) are constant. Obtain the velocity profile \( \mathbf{v} = U_2(y)\mathbf{e}_x \) and the pressure \( p = p_2 \) downstream from the contraction (i.e., as \( x \to +\infty \) for \( 0 \leq y \leq H_2 \)).
For \( U_1 = A \gamma \), we have \( y = \int_0^y (U_1(y)) \, dy \). Setting \( y = \sqrt{y^2 + \frac{2(P_1-P_2)}{\gamma A^2}} \) implies \( y = \sqrt{y^2 + \frac{2(P_1-P_2)}{\gamma A^2}} \), or \( \gamma = \frac{2(P_1-P_2)}{\gamma A^2} \), which leads to \( \gamma = \gamma \left( y + \frac{H_1^2-H_2^2}{H_2} \right) \).

\[
\begin{align*}
U_2^2 &= \sqrt{U_1^2(\gamma_0) + \frac{2P_1-P_2}{\gamma}} \gamma = \sqrt{\gamma_0^2 + A^2 \left( \frac{H_1^2-H_2^2}{2H_2} \right)^2} = A \sqrt{\left( y + \frac{H_1^2-H_2^2}{H_2} \right)^2} = A \left( y + \frac{H_1^2-H_2^2}{H_2} \right)
\end{align*}
\]

The solution can also be obtained by noting that

\[
\text{if } \gamma = 0 \implies \gamma = A \frac{d\gamma}{dy} \frac{dy}{dy}
\]

As \( y \to \infty \), \( \gamma \to \gamma \), we have \( \gamma = A \gamma + \gamma \).

Using the continuity, \( \int_0^y \gamma \, dy = \int_0^x (\gamma(y) \, dy) \to \gamma = A \frac{H_1^2-H_2^2}{B \gamma} \).

\[
\begin{align*}
\frac{P_1-P_2}{\gamma} &= \frac{\gamma^2}{2} = \frac{A^2}{2} \left( \frac{H_1^2-H_2^2}{2H_2} \right)^2
\end{align*}
\]