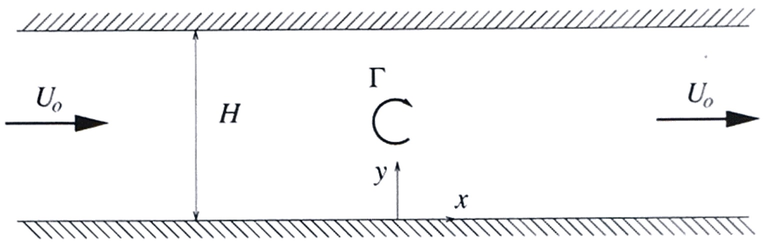
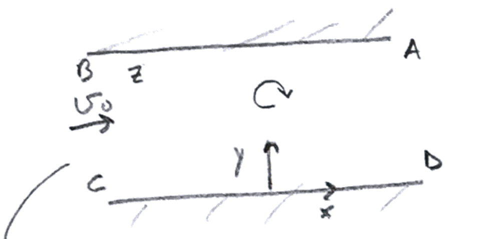
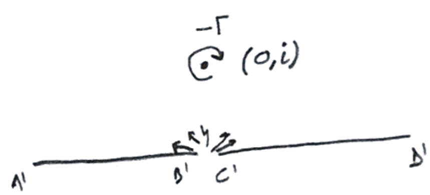
The sketch below represents a model problem to investigate the flow in a wind tunnel of height H with uniform velocity U_o. The presence of the airfoil is represented by a line vortex of negative circulation $-\Gamma$ located at $z = iH/2$. To solve the problem consider the conformal transformation $Z = \exp[\pi z/H]$.

- Obtain the complex potential $f(z)$.
- Determine the velocity and pressure along the upper wall.
- Calculate the location of the stagnation points in terms of $K = \Gamma/(2U_o H)$.
- Represent schematically the streamlines for $K < 1$, $K = 1$, and $K > 1$.







AS $z \to -\infty + iy$, $f(z) = U_0 z$ → AS $Z \to 0$, $F(Z) = \frac{U_0 H}{\pi} \ln(Z)$ → SOURCE OF VOLUME FLOW RATE $Q = 2U_0 H$ AT THE ORIGIN!

$$F(Z) = \frac{\Gamma i}{2\pi}\ln(Z-i) - \frac{\Gamma i}{2\pi}\ln(Z+i) + \frac{U_0 H}{\pi}\ln(Z)$$

$$f(z) = F(Z(z)) = U_0 z + \frac{\Gamma i}{2\pi}\ln\left(\frac{e^{\pi z/H} - i}{e^{\pi z/H} + i}\right)$$

UPPER WALL $z = x + Hi$

$$u - iv = \frac{df}{dz} = U_0 + \frac{\Gamma i}{2\pi}\left(\frac{\frac{\pi}{H}e^{\pi z/H}}{e^{\pi z/H} - i} - \frac{\frac{\pi}{H}e^{\pi z/H}}{e^{\pi z/H} + i}\right) = U_0 + \frac{\Gamma}{H}\frac{e^{\pi x/H}}{1 + e^{2\pi x/H}}$$

$$p - p_\infty = \frac{1}{2}\rho U_0^2\left[1 - \left(1 + \frac{\Gamma}{HU_0}\frac{e^{\pi x/H}}{1 + e^{2\pi x/H}}\right)^2\right]$$

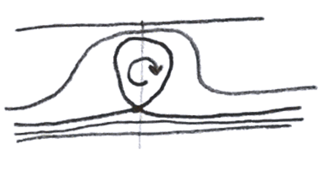
$\frac{df}{dz} = 0 = \frac{dF}{dZ}\frac{dZ}{dz}$ STAGNATION POINTS MAP INTO STAGNATION POINTS

$\frac{dZ}{dz}$ → $e^{\pi z/H} \neq 0$

IN THE Z PLANE → $\frac{dF}{dZ} = \frac{\Gamma i}{2\pi}\left(\frac{1}{Z-i} - \frac{1}{Z+i}\right) + \frac{U_0 H}{\pi Z} = 0$

$$Z = K \pm \sqrt{K^2 - 1} \Rightarrow z = \frac{H}{\pi}\ln\left[K \pm \sqrt{K^2-1}\right]$$

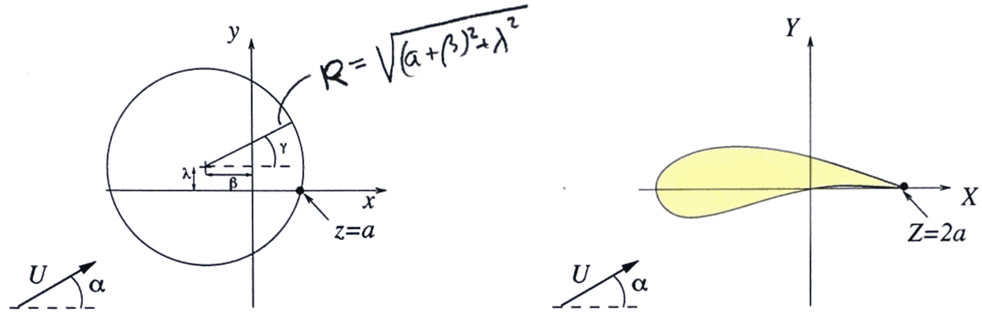
$K < 1$



$K = 1$

$K > 1$

Consider the effect of the Joukowski transformation $Z = z + a^2/z$ on the circle $z = -\beta + i\lambda + [(a+\beta)^2 + \lambda^2]^{1/2} e^{i\gamma}$ with $0 \le \gamma \le 2\pi$, verifying that it maps onto a curved airfoil with finite thickness. Plot the resulting airfoil shape for $\beta/a = 0.5$ and $\lambda/a = 0.5$. Use the transformation to study the flow over the airfoil when flying with angle of attack α. Obtain the complex potential $F(Z)$. Use the Kutta condition to determine the circulation. Simplify the result when $\alpha \ll 1$, $\beta/a \ll 1$, and $\lambda/a \ll 1$. Obtain in that case the lift coefficient.



$$f(z) = U_\infty \left(z e^{-i\alpha} + e^{i\alpha} \frac{(a+\beta)^2 + \lambda^2}{z + \beta - \lambda i} \right) - \frac{\Gamma i}{2\pi} \ln(z + \beta - \lambda i), \quad z = \frac{1}{2} Z + \left(\frac{1}{4} Z^2 - a^2 \right)^{1/2}$$

$$\frac{dF}{dZ} = \frac{df/dz}{dZ/dz}$$

AT $Z = 2a$ $(z=a)$ $\frac{dF}{dZ} \neq \infty$, WHICH IMPLIES THAT $\frac{df}{dz} = 0$ AT $z = a$

$$\frac{df}{dz} = U_\infty \left(e^{-i\alpha} - e^{i\alpha} \frac{(a+\beta)^2 + \lambda^2}{(z + \beta - \lambda i)^2} \right) - \frac{\Gamma i}{2\pi} \frac{1}{z + \beta - \lambda i}$$

$z = a$, $\varphi = \arctan\left(\frac{\lambda}{a+\beta}\right)$, $R = \sqrt{(a+\beta)^2 + \lambda^2}$

$$\frac{df}{dz} = U_\infty \left(e^{-i\alpha} - e^{i\alpha} e^{2\varphi i} \right) - \frac{\Gamma}{2\pi R} e^{\left(\frac{\pi}{2} + \varphi\right) i} = 0 \Rightarrow \boxed{\Gamma = -4\pi U_\infty R \sin(\alpha + \varphi)}$$

$$\varphi \simeq \frac{\lambda}{a}, \; R \simeq a \rightarrow \boxed{\Gamma = -4\pi U_\infty a \left(\alpha + \frac{\lambda}{a} \right)}$$

$$C_L = -\frac{\rho U_\infty \Gamma}{\frac{1}{2} \rho U_\infty^2 (4a)} = 2\pi \left(\alpha + \frac{\lambda}{a} \right)$$

↳ WHEN $\frac{\beta}{a} \ll 1$, $\frac{\lambda}{a} \ll 1$