A layer of liquid of density $\rho$ and viscosity $\mu$ lies on the surface of an infinite flat surface inclined at an angle $\alpha$, as indicated in the figure. Consider the steady motion induced in the presence of gravity when the wall is moving upwards parallel to itself with speed $V$. Obtain the pressure distribution across the layer assuming that the pressure differences in the air are negligibly small, so that $p = p_a$ at $y = h$. Determine the velocity distribution $v_x(y)$ for $0 \leq y \leq h$. Find the viscous force per unit surface on the wall. Calculate the value of $V$ for which the volume flux $\int_0^h v_x \, dy$ is identically zero.

\[
0 = -\frac{\partial p}{\partial y} - \rho g \cos \alpha \Rightarrow p - p_a = \rho g \cos \alpha (h - y)
\]

\[
\begin{align*}
0 &= -\frac{\partial p}{\partial x} + \rho g \sin \alpha + \mu \frac{\partial^2 v_x}{\partial y^2} \\
&\Rightarrow v_x(0) = -V, \quad \frac{\partial v_x}{\partial y} (y = h) = 0
\end{align*}
\]

\[
\begin{align*}
\tau_{xy} &= \mu \frac{\partial v_x}{\partial y} \\
&= \rho g \sin \alpha h
\end{align*}
\]

\[
\int_0^h v_x \, dy = -V h + \frac{\rho g \sin \alpha h^3}{3} = 0 \Rightarrow V = \frac{\rho g h^2 \sin \alpha}{3 \mu}
\]
Consider the unidirectional periodic motion induced in an infinitely long circular pipe of radius \( a \) by an oscillatory pressure gradient \( -\partial p/\partial z = \rho A \cos(\omega t) \). Write the conservation equation with boundary conditions that determine the axial velocity \( v_x(r, t) \) and show how the problem can be solved exactly by separation of variables. Study separately the limits \( a^2 \omega / \nu \gg 1 \) and \( a^2 \omega / \nu \ll 1 \) and obtain the corresponding limiting solutions. In biofluid mechanics the square root \( a \omega / \sqrt{\nu} \) is called the Womersley parameter, which takes fairly large values for blood flow in large arteries, as you can see by using the values corresponding to the human aorta (\( \mu / \rho \approx 4 \times 10^{-2} \) cm²/s, \( a \approx 1.2 \) cm, and \( \omega = 2\pi \) s⁻¹), but that decreases for flow in smaller arteries. Find how small the artery radius needs to be for Poiseuille flow to be approximately applicable.

\[
\frac{\partial v_x}{\partial t} = A \cos(\omega t) + \frac{v_x}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_x}{\partial r} \right)
\]

\( v_x(0) = 0 \)

\( \frac{\partial v_x}{\partial r} (r=\infty) = 0 \) \( (\alpha \quad v_x(r=\infty) \neq 0) \)

\( v_x = Re(\tilde{v}) \) \( \text{where } \tilde{v} \text{ satisfies} \)

\[
\frac{\partial \tilde{v}}{\partial t} = \frac{A e^{i\omega t}}{r} + \frac{\nu}{\partial r} \left( r \frac{\partial \tilde{v}}{\partial r} \right)
\]

\( \tilde{v}(r=\infty) = 0 \)

\( \tilde{v}(r=0) = 0 \)

\[
\tilde{v} = e^{i\omega t} f(r) \rightarrow \quad i \omega f = A + \frac{\nu}{r} \frac{d}{dr} \left( r \frac{df}{dr} \right)
\]

\( f(0) = 0 \)

\( \frac{df}{dr} (r=\infty) = 0 \)

\[
\frac{\partial}{\partial r} \left( \frac{f}{br} \right) = \frac{A}{r} \quad b = \frac{\lambda}{\nu} \frac{J_0}{J_0} \left( \frac{a}{\nu \omega} \right)
\]

IF \( \frac{a^2 \omega}{\nu} \ll 1 \) → \( v_x = A \cos(\omega t) + \frac{\nu}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_x}{\partial r} \right) \rightarrow \quad v_x = \frac{A \cos(\omega t)}{4} \left( a^2 - r^2 \right) \)

IF \( \frac{a^2 \omega}{\nu} \gg 1 \)

\[
\frac{\partial v_x}{\partial t} = A \cos(\omega t) \Rightarrow v_x = \frac{A}{\omega} \sin(\omega t) \rightarrow \quad \text{UNIFORM PROFILE. IT DOES NOT SATISFY THE NONSLIP CONDITION AT THE WALL}
\]

NEAR THE WALL THERE EXISTS A STOKES LAYER OF THICKNESS \( b = \sqrt{\nu / \omega} \) WHICH CAN BE DESCRIBED BY INTRODUCING \( Y = \frac{a-r}{\sqrt{\nu / \omega}} \), \( U = \frac{a}{\omega} \sin(\omega t) - V_x \), \( \omega t \equiv \tau \)

\[
\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial Y^2} \quad Y = 0 : U = \frac{a}{\omega} \sin(\omega t) \quad Y \rightarrow \infty : U \rightarrow 0
\]

\[
\text{STOKES PROBLEM!!}
\]

IN THE HUMAN AORTA \( \frac{a^2 \omega}{\nu} \approx 226 \), WOMERSLEY IS

VISCOUSITY IS DOMINANT IF \( \frac{a^2 \omega}{\nu} \ll 1 \) \( \Rightarrow a < \frac{e}{\sqrt{\nu \omega}} \approx 0.8 \) mm
A fluid of density \( \rho \) and viscosity \( \mu \) is confined between two parallel walls separated a distance \( b \). The fluid rests on a horizontal surface that moves with velocity \( V e_x \) relative to the vertical walls, inducing a steady unidirectional motion with velocity \( v = v_x(y, z)e_x \). Write the equation with boundary conditions that determines \( v_x(y, z) \). Rewrite the problem in dimensionless form using \( V \) and \( b \) as velocity and length scales. Obtain the solution by separation of variables and show that the volumetric flux is given by \( Q \approx 0.27Vb^2 \).