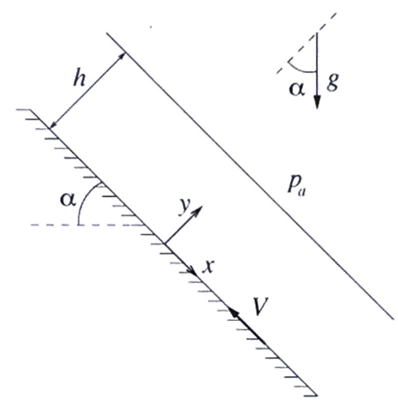
A layer of liquid of density ρ and viscosity μ lies on the surface of an infinite flat surface inclined at an angle α, as indicated in the figure. Consider the steady motion induced in the presence of gravity when the wall is moving upwards parallel to itself with speed V. Obtain the pressure distribution across the layer assuming that the pressure differences in the air are negligibly small, so that $p = p_a$ at $y = h$. Determine the velocity distribution $v_x(y)$ for $0 \leq y \leq h$. Find the viscous force per unit surface on the wall. Calculate the value of V for which the volume flux $\int_0^h v_x dy$ is identically zero.



$$0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha \rightarrow \boxed{p - p_a = \rho g \cos\alpha\,(h-y)}$$

$$V_x(0) = -V;\quad \frac{\partial V_x}{\partial y}(y=h) = 0$$

$$0 = -\cancel{\frac{\partial p}{\partial x}}^{\,0} + \rho g \sin\alpha + \mu \frac{\partial^2 V_x}{\partial y^2} \Rightarrow \boxed{V_x = -V + \frac{\rho g \sin\alpha}{2\mu}\, y(2h-y)}$$

$$\boxed{\tau_w = \mu \frac{\partial V_x}{\partial y}\Big|_{y=0} = \rho g \sin\alpha\, h}$$

$$\int_0^h V_x\, dy = -Vh + \frac{\rho g \sin\alpha}{3\mu} h^3 = 0 \Rightarrow \boxed{V = \frac{\rho g h^2 \sin\alpha}{3\mu}}$$

Consider the unidirectional periodic motion induced in an infinitely long circular pipe of radius a by an oscillatory pressure gradient $-\partial p/\partial x = \rho A\cos(\omega t)$. Write the conservation equation with boundary conditions that determine the axial velocity $v_x(r,t)$ and show how the problem can be solved exactly by separation of variables. Study separately the limits $a^2\omega/\nu \gg 1$ and $a^2\omega/\nu \ll 1$ and obtain the corresponding limiting solutions. In biofluid mechanics the square root $a(\omega/\nu)^{1/2}$ is called the Womersley parameter, which takes fairly large values for blood flow in large arteries, as you can see by using the values corresponding to the human aorta ($\mu/\rho \simeq 4\times 10^{-2}$ cm^2/s, $a \simeq 1.2$ cm, and $\omega = 2\pi$ s^{-1}), but that decreases for flow is smaller arteries. Find how small the artery radius needs to be for Poiseuille flow to be approximately applicable.

$$\frac{\partial V_x}{\partial t} = A\cos(\omega t) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) \qquad V_x(a)=0$$

$$\frac{\partial V_x}{\partial r}(r=0)=0 \quad (\text{or } V_x(r=0)\neq\infty)$$

$V_x = Re(\tilde{V})$ WHERE $\tilde{V}$ SATISFIES

$$\frac{\partial\tilde{V}}{\partial t} = Ae^{i\omega t} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{V}}{\partial r}\right) \qquad \tilde{V}(r=a)=0, \quad \frac{\partial\tilde{V}}{\partial r}(r=0)=0$$

$$\tilde{V} = e^{i\omega t}f(r) \rightarrow \boxed{i\omega f = A + \frac{\nu}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) \quad f(a)=0, \quad \frac{df}{dr}(r=0)=0}$$

$$f = i\frac{A}{\omega}\,\frac{J_0\left(\frac{r}{\sqrt{i\nu/\omega}}\right) - J_0\left(\frac{a}{\sqrt{i\nu/\omega}}\right)}{J_0\left(\frac{a}{\sqrt{i\nu/\omega}}\right)}$$

$$\frac{O\left(\rho\frac{\partial V_x}{\partial t}\right)}{O\left(\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right)\right)} = \frac{a^2\omega}{\nu}$$

IF $\frac{a^2\omega}{\nu} \ll 1 \rightarrow 0 \simeq A\cos(\omega t) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_x}{\partial r}\right) \Rightarrow \boxed{V_x = \frac{A\cos(\omega t)}{4\nu}(a^2-r^2)}$

IF $\frac{a^2\omega}{\nu} \gg 1$ $\quad \frac{\partial V_x}{\partial t} \simeq A\cos(\omega t) \Rightarrow V_x = \frac{A}{\omega}\sin(\omega t) \rightarrow$ UNIFORM PROFILE. IT DOES NOT SATISFY THE NON SLIP CONDITION AT THE WALL

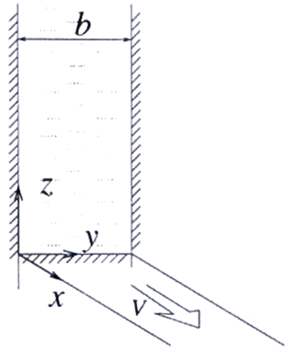
NEAR THE WALL THERE EXISTS A STOKES LAYER OF THICKNESS $\delta = \sqrt{\nu/\omega}$ WHICH CAN BE DESCRIBED BY INTRODUCING $Y = \frac{a-r}{\sqrt{\nu/\omega}}$, $U = \frac{A}{\omega}\sin(\omega t) - V_x$, $\omega t = \tau$

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial Y^2} \quad \left.\begin{array}{l} Y=0:\ U = \frac{A}{\omega}\sin(\omega t) \\ Y\to\infty:\ U\to 0 \end{array}\right\} \text{STOKES PROBLEM!!}$$

IN THE HUMAN AORTA $\frac{a^2\omega}{\nu} \simeq 226$, WOMERSLEY $\simeq 15$

VISCOSITY IS DOMINANT IF $\frac{a^2\omega}{\nu} \ll 1 \Rightarrow \boxed{a \ll \sqrt{\nu/\omega} \simeq 0.8 \text{ mm}}$

A fluid of density ρ and viscosity μ is confined between two parallel walls separated a distance b. The fluid rests on a horizontal surface that moves with velocity $V\mathbf{e}_x$ relative to the vertical walls, inducing a steady unidirectional motion with velocity $\mathbf{v} = v_x(y,z)\mathbf{e}_x$. Write the equation with boundary conditions that determines $v_x(y,z)$. Rewrite the problem in dimensionless form using V and b as velocity and length scales. Obtain the solution by separation of variables and show that the volumetric flux is given by $Q \simeq 0.27Vb^2$.



$0 = -\frac{\partial p}{\partial z} \Rightarrow p = p_\infty, \quad \frac{\partial p}{\partial x} = 0$

$$\boxed{\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = 0 \qquad \begin{array}{l} v_x = 0 \text{ AT } y = 0, b \text{ FOR } z > 0 \\ v_x = 0 \text{ AS } z \to \infty \text{ FOR } 0 \le y \le b \\ v_x = V \text{ AT } z = 0 \text{ FOR } 0 < y < b \end{array}}$$

$u = \frac{v_x}{V}, \quad \eta = y/b, \quad \xi = \frac{z}{b}$

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \xi^2} = 0 \qquad \begin{array}{l} u = 0 \text{ AT } \eta = 0, 1 \text{ FOR } \xi > 0 \\ u = 0 \text{ AS } \xi \to \infty \text{ FOR } 0 \le \eta \le 1 \\ u = 1 \text{ AT } \xi = 0 \text{ FOR } 0 < \eta < 1 \end{array}$$

$$u = f(\eta)g(\xi), \quad \frac{1}{g}\frac{d^2 g}{d\xi^2} = -\frac{1}{f}\frac{d^2 f}{d\eta^2} = \lambda_n^2 \Rightarrow u = \left[C_n \sin(\lambda_n \eta) + D_n \cos(\lambda_n \eta)\right]\left[E_n e^{\lambda_n \xi} + F_n e^{-\lambda_n \xi}\right]$$

$$\left.\begin{array}{l} u = 0 \text{ AT } \eta = 0 \to D_n = 0 \\ u = 0 \text{ AT } \eta = 1 \to \lambda_n = \pi n, \ n = 1, 2, 3 \\ u = 0 \text{ AS } \eta \to \infty \to E_n = 0 \end{array}\right\} \quad u = \sum A_n \sin(n\pi\eta)\, e^{-\pi n \xi}$$

$$1 = \sum A_n \sin(n\pi\eta)$$

$A_n = 0 \quad n = 2, 4, 6 \ldots$

$A_n = \frac{4}{\pi n} \quad \text{FOR } n = 1, 3, 5, \ldots$

$$\frac{Q}{Vb^2} = \int_0^1 \int_0^\infty u \, d\xi \, d\eta = \frac{8}{\pi^3} \sum \frac{1}{n^3} = \frac{8}{\pi^3}\left(1 + \frac{1}{27} + \frac{1}{125} + \cdots\right) \simeq 0.27$$