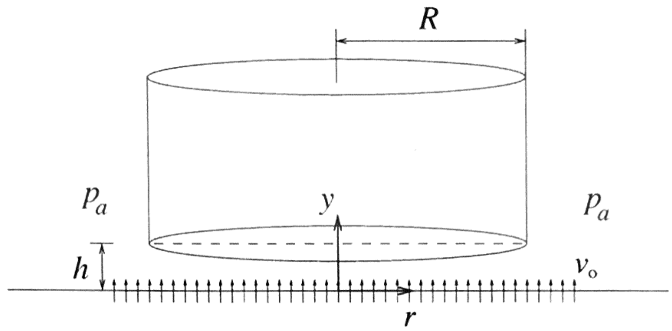
Consider a solid cylinder of mass M and radius R located over a planar porous surface as indicated in the figure below. Gas is injected normal to the surface with velocity v_o, inducing overpressures $p - p_a$ that allow the cylinder to levitate at a small distance $h \ll R$. Give the criterion needed for the flow to be dominated by viscosity. Obtain the pressure distribution $p(r) - p_a$ as well as the force acting on the cylinder and the height h.



$\boxed{v_y \sim v_o}$, $\quad \frac{\partial v_y}{\partial y} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0 \quad \boxed{v_r \sim v_o \frac{R}{h}} \quad \frac{O(\rho \bar{v}\cdot\nabla\bar{v})}{O(\mu \nabla^2 \bar{v})} \sim \frac{\rho v_r h}{\mu}\frac{h}{R} \ll 1 \Rightarrow \boxed{\frac{\rho v_o h}{\mu} \ll 1}$

$$0 = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 v_r}{\partial y^2} \Rightarrow v_r = -\frac{1}{2\mu}\frac{\partial p}{\partial r} y(h-y) \rightarrow \int_0^h v_r \, dy = \frac{-h^3}{12\mu}\frac{dp}{dr}$$

$$\int_0^h \frac{\partial}{\partial y}(r v_y)\, dy + \int_0^h \frac{\partial}{\partial r}(r v_r)\, dy = 0 \rightarrow -r v_o + \frac{d}{dr}\left(r \int_0^h v_r \, dy\right) = 0 \Rightarrow \frac{d}{dr}\left(r \frac{dp}{dr}\right) = \frac{-12\mu v_o}{h^3} r$$

$$\downarrow$$

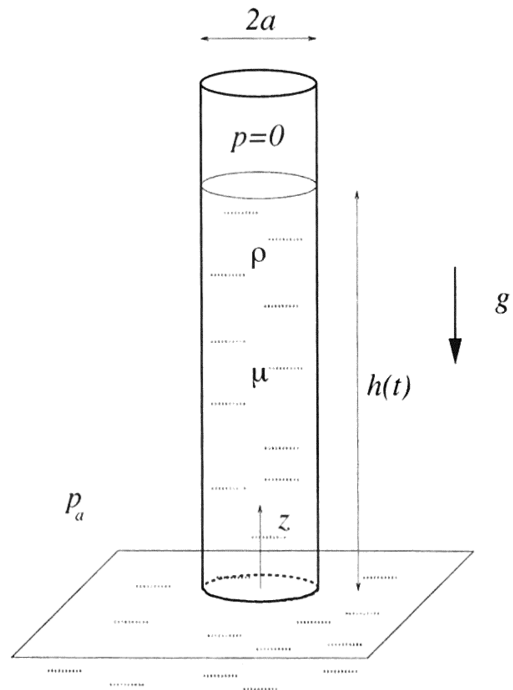
$$r\frac{dp}{dr} = \frac{-6\mu v_o}{h^3} r^2 + \cancel{C_1}$$

$$\boxed{p - p_a = \frac{3\mu v_o}{h^3}(R^2 - r^2)}$$

$$\boxed{\bar{F} = \bar{e}_y \int_0^R (p - p_a)\, 2\pi r \, dr = \frac{3}{2}\pi \frac{\mu v_o R^4}{h^3}\bar{e}_y} \qquad F = Mg \Rightarrow \boxed{h = \left(\frac{3}{2}\pi \frac{\mu v_o R^4}{Mg}\right)^{1/3}}$$

The bottom end of an empty vertical tube of radius a, closed at the top, is put in contact with a pool of oil. Because of the ambient overpressure, the liquid begins to flow into the tube, forming a column of increasing height $h(t)$ whose evolution in time is to be investigated assuming that the motion is dominated by viscosity. In particular:

- Obtain the value of the height h_∞ corresponding to the equilibrium position, reached asymptotically for large times.
- Give the condition that determines whether the motion is dominated by viscosity.
- Obtain the evolution of $h(t)$ as well as the pressure distribution along the pipe $p(x,t)$ for $0 < x < h$.
- Compute the force acting on the pipe as a function of time $\bar{F} = F_z(t)\bar{e}_z$.



1) FOR THE FINAL EQUILIBRIUM POSITION, THE HYDROSTATIC PRESSURE DISTRIBUTION ALONG THE COLUMN IS GIVEN BY

$$P + \rho g z = P_a = \rho g h_\infty \Rightarrow \boxed{h_\infty = P_a/\rho g}$$

2) IF THE TRANSIENT IS DOMINATED BY VISCOUS FORCES $\rightarrow$ $0 = -\nabla P + \mu \nabla^2 \bar{v}$ $\Rightarrow$ $V_c \sim \frac{a^2}{h_\infty}\frac{P_a}{\mu} \sim \frac{\rho g a^2}{\mu}$

$$P_a/h_\infty \sim \mu \frac{V_c}{a^2}$$

$$\frac{O(\rho\, \partial \bar{v}/\partial t)}{O(\mu \nabla^2 \bar{v})} \sim \frac{a^2/(\mu/\rho)}{h_\infty/V_c} \ll 1 \rightarrow \boxed{\frac{\rho^3 g^2 a^4}{\mu^2 P_a} \ll 1}$$

$$\frac{O(\rho \bar{v}\cdot\nabla \bar{v})}{O(\mu \nabla^2 \bar{v})} \sim \frac{\rho V_c a}{\mu}\frac{a}{h_\infty} \ll 1.$$

$$\int_0^a \left[\frac{\partial}{\partial z}(r v_z) + \frac{\partial}{\partial r}(r v_r)\right] dr = 0 \qquad \frac{\partial Q}{\partial z} = 0 \rightarrow Q = Q(t)$$

3) $$0 = -\frac{\partial (p + \rho g z)}{\partial z} + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) \xrightarrow{r=a,\ v_z=0} v_z = -\frac{1}{4\mu}\frac{\partial P}{\partial z}(a^2 - r^2) \rightarrow Q(t) = \int_0^a 2\pi v_z r\, dr = -\frac{\pi}{8\mu}\frac{\partial P}{\partial z}a^4$$

($P \to p + \rho g z$)

$$z=0,\ P=P_a \rightarrow \boxed{P + \rho g z = P_a - \frac{8\mu Q}{\pi a^2} z}$$

AT $x = h$, $P=0$ $\Rightarrow$ $\rho g h = P_a - \frac{8\mu Q}{\pi a^2} h$

USING $Q = \pi a^2 \frac{dh}{dt}$ $\rightarrow$ $\boxed{\frac{dh}{dt} = \frac{P_a}{8\mu}\frac{1 - h/h_\infty}{h}, \ h(0) = 0}$

$$\boxed{\frac{P_a}{8\mu h_\infty^2} t = \ln\left(\frac{1}{1 - h/h_\infty}\right) - h/h_\infty}$$

4)

$$\tau_w = -\mu \frac{\partial V_x}{\partial r} = \frac{a}{2}\frac{P_a - \rho g h}{h}$$

$$F_w = 2\pi a h \tau_w = \pi a^2 (P_a - \rho g h)$$

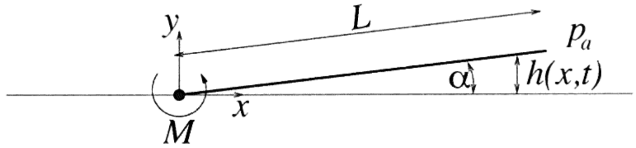
TOTAL FORCE $F_w - \underbrace{P_a \pi a^2}_{\text{AT THE TOP END}} = -\rho g h \pi a^2 \bar{e}_z$

A plate of length L is initially sitting on a horizontal plane surrounded by a stagnant atmosphere at pressure p_a. At a given instant, we begin to rotate the plate with constant angular velocity $\Omega = d\alpha/dt$ by applying a given torque M at its left end, as sketched in the figure. For the analysis, use the approximation $h(x,t) = x\tan(\alpha) \simeq x\alpha$, valid for $\alpha \ll 1$.

1. Demonstrate that for values of α sufficiently smaller than a critical value, to be determined, the fluid motion in the gap formed between the plate and the wall is dominated by viscosity.
2. Obtain the velocity profile v_x in the gap as a function of the unknown value of $P_l(x,t) = -\partial p/\partial x$ as well as the associated volume flux at a given section $Q = \int_0^{\alpha x} v_x dy$.
3. Using continuity, write an equation linking Q and Ω, and integrate it to compute the pressure distribution $p(x,t)$ (in the integration, you may anticipate that $x^3 P_l \to 0$ as $x \to 0$).
4. Determine the torque $M(t)$ needed to provide a constant angular velocity Ω.

$h_c \sim L\alpha$

$\Omega = \frac{d\alpha}{dt} \Rightarrow t_o \sim \frac{\alpha}{\Omega}$



$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \quad \left\{ \begin{array}{l} V_y \sim \Omega L \\ V_x \sim \frac{\Omega L^2}{h_c} \end{array} \right.$$

$$\frac{V_x}{L} \sim \frac{V_y}{h_c}$$

1) $$\frac{O(\rho\, \partial \bar{v}/\partial t)}{O(\mu \nabla^2 \bar{v})} \sim \frac{h_c^2/(\mu/\rho)}{t_o} \sim \frac{L^2\alpha^2\Omega}{\nu\alpha} \ll 1 \Rightarrow \boxed{\alpha \ll \frac{\nu}{L^2\Omega}}$$

$$\frac{O(\rho \bar{v}\cdot\nabla\bar{v})}{O(\mu\nabla^2\bar{v})} \sim \frac{V_x h_c}{\nu}\frac{h_c}{L} \sim \frac{\Omega L^2 \alpha}{\nu} \ll 1$$

2) $$V_x = \frac{P_l}{2\mu} y(h-y) \rightarrow Q = \int_0^{\alpha x} V_x dy = -\frac{x^3\alpha^3}{12\mu}\frac{\partial p}{\partial x}$$

3) $$\int_0^h \frac{\partial V_x}{\partial x} dy + \int_0^h \frac{\partial V_y}{\partial y} dy = 0 \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial h}{\partial x}\cancel{V_x(y=h)} + V_y(h) - \cancel{V_y(y=0)} = 0$$

$$\frac{\partial Q}{\partial x} + \Omega x = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\alpha^3}{12\mu} x^3 \frac{\partial p}{\partial x}\right) - \Omega x = 0 \Rightarrow \boxed{p - p_a = \frac{6\mu\Omega}{\alpha^3}\ln\left(\frac{x}{L}\right)}$$

4) $$M = +\int_0^L (p - p_a) x\, dx = -\frac{3}{2}\frac{\mu\Omega L^2}{\alpha^3}$$

TO COUNTERBALANCE THIS MOMENT WE NEED TO APPLY A POSITIVE (COUNTERCLOCKWISE) TORQUE $M = +\frac{3}{2}\frac{\mu\Omega L^2}{\alpha^3}$