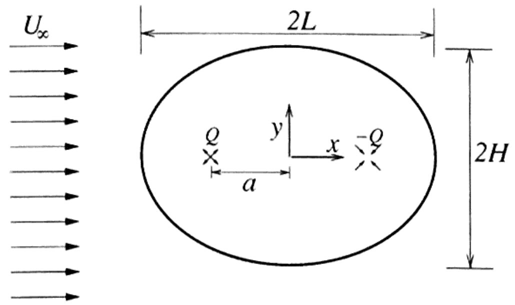
A uniform stream of velocity U_∞ flows past a two-dimensional symmetric body of length $2L$ and height $2H$. An approximate solution can be generated by superimposing on the free stream a source of strength Q located at $z = -a$ and a sink of strength $-Q$ located at $z = a$, as sketched in the figure. To analyze the flow, follow these steps:

1. Write the complex potential.
2. Obtain the stagnation points and use the resulting equation to relate the relative semi-length L/a with the parameter $\Lambda = Q/(\pi U_\infty a)$.
3. Determine the stream function as well as the equation for the stream line that defines the contour of the body.
4. Write an equation relating the relative semi-height H/a with the parameter Λ.
5. For $H/a = 2$, obtain the corresponding values of Λ and L/a.
6. For this last case, compute the maximum velocity over the body, achieved at $z = \pm Hi$.



1) $f(z) = U_\infty z + \frac{Q}{2\pi}\ln(z+a) - \frac{Q}{2\pi}\ln(z-a)$

2) $\frac{df}{dz} = u - iv = U_\infty - \frac{Q}{\pi a}\frac{1}{(z/a)^2-1}$, $\frac{df}{dz} = 0 \Rightarrow \frac{z}{a} = \pm(1+\Lambda)^{1/2}$

$$\boxed{\frac{L}{a} = (1+\Lambda)^{1/2}}$$

3) $\psi = \mathrm{Im}(f) = U_\infty y - \frac{Q}{2\pi}\arctan\left(\frac{2ay}{x^2+y^2-a^2}\right)$, THE LIMITING STREAMLINE IS

$$\boxed{\frac{y}{a} = \frac{\Lambda}{2}\arctan\left(\frac{2\,y/a}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2-1}\right)}$$

4) $\frac{H}{a} = \frac{\Lambda}{2}\arctan\left(\frac{2\,H/a}{\left(\frac{H}{a}\right)^2-1}\right)$

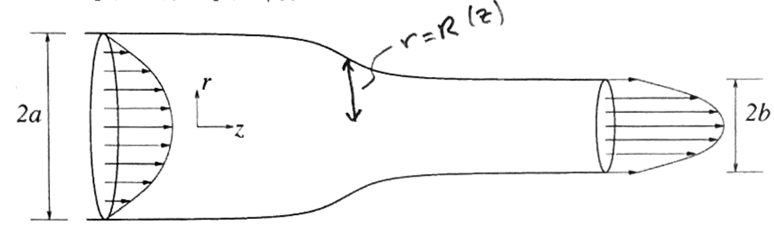
5) e.g. For $H/a = 2$

$$\Lambda = \frac{2\,H/a}{\arctan\left(\frac{2\,H/a}{\left(\frac{H}{a}\right)^2-1}\right)} = 4.31 , \quad \frac{L}{a} = 2.305$$

6) $|\bar{V}|_{max} = U_\infty\left(1+\frac{\Lambda}{\left(\frac{H}{a}\right)^2+1}\right) = 1.862$

Consider the steady axisymmetric flow of a fluid of constant density ρ and constant viscosity μ in the contraction shown in the figure, which connects two coaxial cylindrical pipes of radii a and $b < a$. Upstream from the contraction the velocity is given by the Poiseuille profile $v_z = U[1-(r/a)^2]$, where U is the maximum velocity. Assuming that $\rho U a/\mu \sim \rho U b/\mu \gg 1$:

- Write the equations with boundary conditions that determine the velocity and pressure fields.
- Show that the vorticity magnitude $\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$ is given by $\omega_\theta = 2Ur/a^2$ everywhere in the flow field.
- Obtain the velocity profile $v_z(r)$ downstream from the contraction, as well as the total pressure drop across the contraction $p_{z\to-\infty} - p_{z\to+\infty}$.



STEADY AXISYMMETRIC INVISCID FLOW

$\nabla\cdot\bar{v} = 0 \Rightarrow \frac{\partial v_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0$

$\bar{v}\cdot\nabla\bar{v} = -\nabla(p/\rho) \Rightarrow v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} = -\frac{\partial p/\rho}{\partial r}$

$v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z}$

$z\to-\infty$, $v_r = 0$, $v_z = U\left[1-(r/a)^2\right]$

$z\to+\infty$, $v_r = 0$

$r = R(z)$; $v_r + \frac{dR}{dz}v_z = 0$

SINCE $\bar{v}\cdot\nabla\left(\frac{\omega_\theta}{r}\right) = 0$, $\frac{\omega_\theta}{r}$ IS CONSTANT ALONG EACH STREAMLINE. FOR THE INCOMING FLOW,

$\omega_\theta = -\frac{\partial v_z}{\partial r} = 2U\frac{r}{a^2}$, SO THAT $\frac{\omega_\theta}{r} = \frac{1}{r}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) = \frac{2U}{a^2}$ EVERYWHERE.

AS $z\to+\infty$, $v_r\to 0$ $\to$ $\frac{\partial v_z}{\partial r} = -\frac{2U}{a^2}r \to v_z = \left(C - \frac{r^2}{a^2}\right)U$

ON THE OTHER HAND, FROM CONTINUITY

$$\frac{d}{dz}\int_0^R v_z r\,dr = 0 \Rightarrow \int_0^R v_z r\,dr = Ua^2\int_0^a\left[1-\left(\frac{r}{a}\right)^2\right]\left(\frac{r}{a}\right)d\left(\frac{r}{a}\right) = \frac{Ua^2}{4}$$

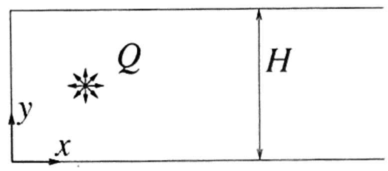
$$\frac{Ua^2}{4} = U\int_0^b\left(C - \frac{r^2}{a^2}\right)r\,dr = U\left[C\frac{b^2}{2} - \frac{b^4}{4a^2}\right] \Rightarrow C = \frac{1}{2}\,\frac{1+(b/a)^4}{(b/a)^2}$$

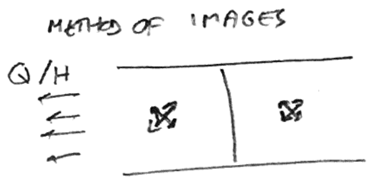
$$v_z = U\left[\frac{1+(b/a)^4}{2(b/a)^2} - \frac{r^2}{a^2}\right]$$

BERNOULLI'S EQ: $p + \frac{1}{2}\rho|\bar{v}|^2 = C\ell$. ALONG AXIS: $p_{-\infty} + \frac{1}{2}\rho U^2 = p_{+\infty} + \frac{1}{2}\rho U^2 C^2 \to$

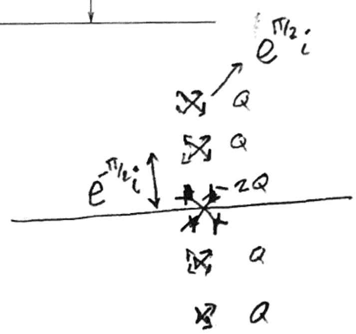
$$\boxed{p_{-\infty} - p_{+\infty} = \frac{1}{2}\rho U^2(C^2-1) = \frac{1}{2}\rho U^2\left(\frac{1-(b/a)^4}{2(b/a)^2}\right)^2}$$

A point source of strength Q is located at $(x, y) = (H/2, H/2)$ inside a semiinfinite channel of thickness H, as indicated in the figure. Determine the complex potential $f(z)$ as well as the pressure along the vertical wall $p(0, y)$ for $0 \le y \le H$.





$$Z = e^{\pi z/H} \Rightarrow$$



$$F(Z) = \frac{Q}{2\pi}\Big[\ln(Z - e^{\pi/2\, i}) + \ln(Z - e^{-\pi/2\, i}) - 2\ln(Z) + \ln(Z + e^{\pi/2\, i}) + \ln(Z + e^{-\pi/2\, i})\Big]$$

$$= \frac{Q}{2\pi}\ln\left[(Z^2 + e^{\pi})(Z^2 + e^{-\pi})Z^{-2}\right]$$

$$\boxed{f(z) = F(Z(z)) = \frac{Q}{2\pi}\ln\left[\left(e^{\frac{2\pi z}{H}} + e^{\pi}\right)\left(e^{\frac{2\pi z}{H}} + e^{-\pi}\right)e^{-\frac{2\pi z}{H}}\right]}$$

$$\frac{df}{dz} = \frac{dF}{dZ}\frac{dZ}{dz} = \frac{Q}{H}\left[\frac{Z^2}{Z^2 + e^{\pi}} + \frac{Z^2}{Z^2 + e^{-\pi}} - 1\right]$$

$$\boxed{z = y\,i} \rightarrow Z = e^{\frac{\pi y}{H} i}, \quad u - iv = \frac{Q}{H}\,\frac{i\sin(2\pi y/H)}{\frac{e^{\pi} + e^{-\pi}}{2} + \cos\left(\frac{2\pi y}{H}\right)}$$

$$u = 0, \quad \boxed{v = -\frac{Q}{H}\,\frac{\sin(2\pi y/H)}{\cosh(\pi) + \cos\left(\frac{2\pi y}{H}\right)}}$$

$$p_\infty + \frac{1}{2}\rho\left(\frac{Q}{H}\right)^2 = p_w + \frac{1}{2}\rho\left(\frac{Q}{H}\right)^2\left(\frac{\sin(2\pi y/H)}{\cosh(\pi) + \cos(2\pi y/H)}\right)^2$$

$$\boxed{p_w - p_\infty = \frac{1}{2}\rho\left(\frac{Q}{H}\right)^2\left[1 - \left(\frac{\sin(2\pi y/H)}{\cosh(\pi) + \cos(2\pi y/H)}\right)^2\right]}$$