Consider the planar flow with velocity components
\[ v_r = U \cos(\theta) \left[ 1 - \left( \frac{a}{r} \right)^2 \right] \quad \text{and} \quad v_\theta = -U \sin(\theta) \left[ 1 + \left( \frac{a}{r} \right)^2 \right] \]

1. Verify that the motion corresponds to that of an incompressible fluid, so that \( \nabla \cdot \mathbf{v} = 0 \).
2. Does the stream function \( \psi(r, \theta) \) exist? If so, compute it.
3. Determine the vorticity \( \omega(r, \theta) = \nabla \wedge \mathbf{v} \).
4. Is the flow irrotational? If so, compute the velocity potential \( \phi(r, \theta) \).
5. Represent schematically the streamlines and show that the circle \( r = a \) is a streamline of the flow.

1) \[ \frac{1}{r} \frac{d}{dr} \left( r v_r \right) + \frac{1}{r} \frac{d}{d\theta} v_\theta = 0 \]

2) \( v_r = \frac{1}{r} \frac{d\psi}{d\theta} \quad \Rightarrow \quad \psi = U r \sin(\theta) \left[ 1 - \left( \frac{a}{r} \right)^2 \right] + f(r) \)

   \( v_\theta = -\frac{1}{r} \frac{d\psi}{dr} \Rightarrow \frac{df}{dr} = 0 \Rightarrow f = \text{const} \quad \psi \quad \text{if we choose} \quad \psi_0 = 0 \Rightarrow \psi = U r \sin(\theta) \left[ 1 - \left( \frac{a}{r} \right)^2 \right] \]

3) \[ \mathbf{\tilde{\omega}} = -\frac{1}{r} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ \frac{1}{r} \frac{d}{dr} & \frac{1}{r} \frac{d}{d\theta} & \frac{1}{r} \frac{d}{d\phi} \\ v_r & v_\theta & v_\phi \end{vmatrix} = \frac{1}{r} \left[ \frac{\partial v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta} \right] = 0 \]

4) \( v_r = \frac{d\phi}{dr} \Rightarrow \phi = U \cos(\theta) \left[ 1 + \left( \frac{a}{r} \right)^2 \right] + g(\theta) \)

   \( v_\theta = \frac{1}{r} \frac{d\phi}{d\theta} \Rightarrow \frac{dg}{d\theta} = 0 \Rightarrow g = \text{const} = \phi \quad \text{if we choose} \quad \phi_0 = 0 \Rightarrow \phi = U r \cos(\theta) \left[ 1 + \left( \frac{a}{r} \right)^2 \right] \]

5) Streamlines \( \psi = \sqrt{U r \sin(\theta) \left[ 1 - \left( \frac{a}{r} \right)^2 \right]} = \text{const} \)

   THE CIRCLE IS THE STREAMLINE \( \psi = 0 \)

\[ \psi = 0 \quad \psi = 0 \quad \psi = 0 \]

\[ \psi = 0 \]
Consider the flow induced by the superposition of a line source of volume rate $Q$ located at $z = 1$, a line sink of volume rate $-Q$ located at $z = -1$, a line vortex of positive circulation $\Gamma$ located at $z = i$, and a line vortex of negative circulation $-\Gamma$ located at $z = -i$. Determine the complex potential and the complex velocity. Determine the location of the stagnation points. Sketch the streamlines for the cases $Q/\Gamma < 1$, $Q/\Gamma = 1$, and $Q/\Gamma > 1$. 

\[ w = \frac{Q}{2\pi} \ln(z-1) = \frac{Q}{2\pi} \ln(z+i) - \frac{\Gamma}{2\pi} \ln(z-i) + \frac{\Gamma}{2\pi} \ln(z+i) \]

\[ u - i v = \frac{dw}{dt} = \frac{\Gamma}{\pi} \frac{(Q\Gamma + 1)z^2 - (1 - Q\Gamma)}{z^2 - 1} \]

$u - iv = 0 \rightarrow \frac{Q}{\Gamma} \leq 1 \rightarrow z = \pm \sqrt{1 - Q\Gamma}$

$\frac{Q}{\Gamma} = 1 \rightarrow z = 0$

$\frac{Q}{\Gamma} > 1 \rightarrow z = \pm i \sqrt{Q\Gamma - 1}$
Consider the flow induced by a source of strength $Q$ located at $z = Re^{i\beta}$ in the corner region of angle $\alpha > \beta$ shown in the figure below.

1. Show that the conformal transformation $Z = (z/R)^{\pi/\alpha}$ maps the corner region to the upper-half $Z$-plane. Show in particular that the two walls $z = r$ and $z = re^{i\alpha}$ become, respectively, the positive and negative real axes and that the source location in the $Z$ plane is $Z = e^{i\Lambda}$, with $\Lambda = \pi \beta / \alpha$.

2. Compute the complex potential $W(Z)$ and the associated complex velocity $dW/dZ$.

3. Determine the stagnation points in the $Z$-plane and their corresponding location in the original $z$-plane.

4. Sketch the streamlines in the $z$-plane when $\beta < \alpha / 2$ and when $\beta > \alpha / 2$.

5. Calculate the pressure distribution on the upper wall, writing the result in the form $(p-p_\infty) / [\rho Q(\alpha R)^2]$, as a function of $\Lambda$ and $r/R$. 

\[ Z = e^{i\beta} \rightarrow Z = e^{i\beta} R \]

\[ W = Q \frac{\alpha}{2\pi} \left[ (Z - e^{i\beta})(Z - e^{-i\beta}) \right] \]

\[ \frac{dW}{dZ} = \frac{Q}{2\pi} \frac{Z(Z - \cos\Lambda)}{Z^2 - 2 \cos\Lambda Z + 1} \]

\[ \text{Stagnation point} \quad Z = \cos\Lambda \rightarrow \left( \frac{Z}{\Lambda} \right) = \cos\Lambda \quad \Rightarrow \quad \cos\Lambda > 0 \Rightarrow Z = R(\cos\Lambda) \]

\[ \frac{dW}{dZ} = \frac{Q}{2\pi} \left( \frac{Z - \cos\Lambda}{Z^2 - 2 \cos\Lambda Z + 1} - \frac{1}{R^2} \right) \]

\[ \left( \frac{r}{R} \right)^{\pi/\alpha} + \cos\Lambda \left( \frac{r}{R} \right)^{\pi/\alpha - 1} \]

\[ \frac{p - p_\infty}{\rho Q(\alpha R)^2} = \left( \frac{\frac{r}{R} + \cos\Lambda}{\left( \frac{r}{R} \right)^{\pi/\alpha} + 2 \cos\Lambda \left( \frac{r}{R} \right)^{\pi/\alpha - 1}} \right)^2 \]