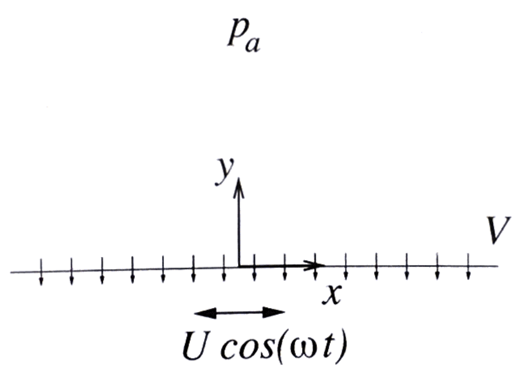
Consider the modified Stokes problem defined in the figure below. The oscillating wall, moving parallel to itself with velocity $U\cos(\omega t)$, is a porous surface through which the fluid is entrained with constant normal velocity $-V$.

1. Anticipating that the x component of the velocity $u(y,t)$ is only a function of the distance to the wall y and the time t, obtain the distribution of transverse velocity $v(x,y,t)$.
2. Assuming that the pressure takes the uniform constant value $p = p_a$ far from the wall, compute the pressure distribution $p(x,y,t)$.
3. Write the equation with boundary conditions that determine $u(y,t)$.
4. Consider the limit $V \gg (\nu\omega)^{1/2}$, demonstrating that the motion is quasi-steady. Find the associated solution for $u(y,t)$.
5. Consider the limit $V \ll (\nu\omega)^{1/2}$, demonstrating that the effect of convection is negligible in the first approximation. Find the associated solution for $u(y,t)$.
6. In the general case, $V \sim (\nu\omega)^{1/2}$ write the problem in dimensionless form with use made of the scales U, ω^{-1}, and ν/V for u, t, and y, respectively. Obtain the exact solution for u/U, verifying that the limiting solutions determined above are recovered in the corresponding limits $V \gg (\nu\omega)^{1/2}$ and $V \ll (\nu\omega)^{1/2}$, respectively.



1) $\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} = 0 \rightarrow v = v(x,t)$ AT $y=0$, $v=-V \Rightarrow \boxed{v(x,t) = -V}$

2) $\rho\cancel{\frac{\partial v}{\partial t}} + \cancel{\rho u \frac{\partial v}{\partial x}} + \cancel{\rho v \frac{\partial v}{\partial y}} = -\frac{\partial p}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right) \rightarrow p = p(x,t)$ AT $y=\infty$, $p = p_a \rightarrow \boxed{p(x,t) = p_a}$

3) $\frac{\partial u}{\partial t} - V\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$ $\begin{cases} y=0: u = U\cos(\omega t) \\ y\to\infty: u\to 0 \end{cases}$

4) THE DOMINANT BALANCES IN THE MOMENTUM EQUATION INDICATE THAT THERE EXIST TWO DIFFERENT CHARACTERISTIC LENGTHS IN THE PROBLEM

$\frac{\partial u}{\partial t} \sim \nu\frac{\partial^2 u}{\partial y^2} \rightarrow \delta_1 = \sqrt{\nu/\omega}$ STOKES THICKNESS

$-V\frac{\partial u}{\partial y} \sim \nu\frac{\partial^2 u}{\partial y^2} \rightarrow \delta_2 = \nu/V$ CONVECTION-DIFFUSION BALANCE

$V \gg (\nu\omega)^{1/2}$ $(\delta_2 \ll \delta_1)$ BECAUSE OF CONVECTION, MOTION IS RESTRICTED TO A THIN LAYER OF THICKNESS $y \sim \frac{\nu}{V} \ll \sqrt{\nu/\omega}$

WHERE $\frac{O(\partial u/\partial t)}{O(\nu\,\partial^2 u/\partial y^2)} \sim \frac{\nu\omega}{V^2} \ll 1 \Rightarrow -V\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \Rightarrow \boxed{u = U\cos(\omega t)\, e^{-Vy/\nu}}$

5) $V \ll (\nu\omega)^{1/2}$ THE REYNOLDS # IN THE STOKES LAYER IS $\frac{V\delta_1}{\nu} = \frac{V}{\sqrt{\nu\omega}} \ll 1$, SO THAT CONVECTION IS NEGLIGIBLE

$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \Rightarrow \boxed{u = U e^{-\sqrt{\frac{\omega}{2\nu}}\,y}\cos\left(\omega t - \sqrt{\frac{\omega}{2\nu}}\,y\right)}$

6) $\bar{u} = \frac{u}{U}$, $\tau = \omega t$, $\eta = \frac{Vy}{\nu}$ $\boxed{\alpha = \frac{\omega\nu}{V^2}}$ $\boxed{\alpha\frac{\partial\bar{u}}{\partial\tau} - \frac{\partial\bar{u}}{\partial\eta} = \frac{\partial^2\bar{u}}{\partial\eta^2} \quad \begin{cases} \eta=0: \bar{u} = \cos\tau \\ \eta\to\infty: \bar{u} = 0 \end{cases}}$

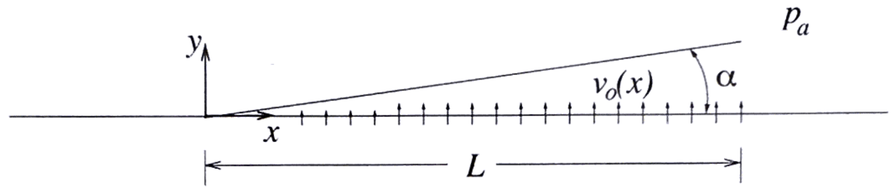
$\bar{u} = \mathrm{Re}\left(e^{i\tau - \frac{1+\sqrt{1+4\alpha i}}{2}\eta}\right)$

$\alpha \ll 0 \longrightarrow \boxed{\bar{u} = e^{-\eta}\cos(\tau)}$

$\alpha \gg 1: \rightarrow \boxed{\bar{u} = e^{-\sqrt{\frac{\alpha}{2}}\,\eta}\cos\left(\tau - \sqrt{\frac{\alpha}{2}}\,\eta\right)}$

A fluid of density ρ and viscosity μ is injected with normal velocity $v_o(x)$ into a channel of length L bounded by two flat walls forming an angle $\alpha \ll 1$, as indicated in the figure. For the steady flow established in the channel:

1. Give the condition for the motion to be dominated by viscous forces.
2. Obtain the pressure distribution along the channel $p(x) - p_a$.
3. For the particular case $v_o = V(x/L)^2$, where V is a constant, obtain the force acting on the upper wall of the channel as well as the moment exerted with respect to the left end $x = 0$.



1) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow U_c \sim \frac{V_c}{\alpha}$ ← CHARACTERISTIC VALUE OF $V_0(x)$

$$\frac{O(\rho \bar{v}\cdot\nabla\bar{v})}{O(\mu\nabla^2\bar{v})} \sim \frac{\rho U_c h}{\mu}\frac{h}{L} \sim \boxed{\frac{\rho V_c L}{\mu}\alpha \ll 1}$$

2) $0 = -\frac{dP}{dx} + \mu\frac{\partial^2 u}{\partial y^2}$, $y=0, \alpha x: u=0 \Rightarrow u = -\frac{1}{2\mu}\frac{dP}{dx}y(\alpha x - y)$, $\int_0^{\alpha x} u\,dy = -\frac{(\alpha x)^3}{12\mu}\frac{dP}{dx}$

$$\int_0^{\alpha x}\frac{\partial u}{\partial x}dy + \int_0^{\alpha x}\frac{\partial v}{\partial y}dy = 0 \rightarrow \frac{d}{dx}\left(\int_0^{\alpha x} u\,dy\right) - V_0(x) = 0 \Rightarrow \frac{d}{dx}\left(\frac{(\alpha x)^3}{12\mu}\frac{dP}{dx}\right) = -V_0(x)$$

$$\frac{(\alpha x)^3}{12\mu}\frac{dP}{dx} = -\int_0^x V_0(x)\,dx$$

$$P - P_a = 12\mu\int_x^L \frac{1}{(\alpha x)^3}\left(\int_0^x V_0\,d\bar{x}\right)dx$$

3) $V_0 = V\left(\frac{x}{L}\right)^2$ $\quad \boxed{P - P_a = \frac{4\mu V}{\alpha^3 L}\left(1 - \frac{x}{L}\right)}$

NORMAL FORCE $\quad \int_0^L (P - P_a)\,dx = \frac{2\mu V}{\alpha^3}$

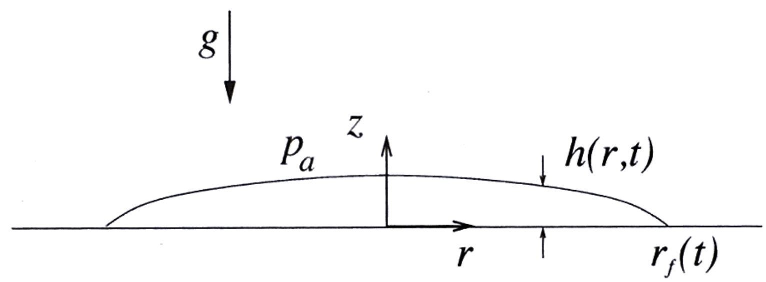
TANGENTIAL FORCE $\quad \int_0^L \tau_w\,dx = \frac{\mu V}{\alpha^2}$

$\quad \tau_w = -\mu\left.\frac{\partial u}{\partial y}\right|_{y=h} = \frac{2\mu V}{\alpha^2 L}\frac{x}{L}$

TORQUE $\quad \int_0^L (P - P_a)\,x\,dx = \frac{2}{3}\frac{\mu V L}{\alpha^3}$

An **axisymmetric** liquid pool of kinematic viscosity ν spreads over a flat horizontal surface under the action of gravity. Initially the pool has radius r_o and thickness distribution $h_o(r)$, such that $V = \int_0^{r_o} 2\pi h_o r dr$ is the total volume of liquid.

1. Give the condition for the motion to be dominated by viscous forces.
2. Write the equation with initial and boundary conditions that determine the thickness distribution $h(r,t)$ as well as the radial location of the pool edge $r_f(t)$.
3. Rewrite the problem in dimensionless form by introduction of the scales r_o, V/r_o^2, and $(\nu/g)(r_o^8/V^3)$ for r, h, and t, respectively.
4. Consider the evolution for $t \gg (\nu/g)(r_o^8/V^3)$. Obtain the characteristic values $h_\infty(t) \ll V/r_o^2$ and $r_\infty(t) \gg r_o$ of h and r_f in this limit.
5. Determine the self-similar solution for the long-time evolution by rewriting the problem in terms of the rescaled variables $H(\eta) = h/h_\infty(t)$ and $\eta = r/r_\infty(t)$. Obtain in particular an expression for the pool-edge location $r_f(t)$ for $t \gg (\nu/g)(r_o^8/V^3)$.



1) IF DOMINATED BY VISCOUS FORCES $\Delta P \sim \rho g h_o \sim \rho g \frac{V}{r_o^2}$, $-\frac{\partial P}{\partial r} \sim \mu \frac{\partial^2 V_r}{\partial z^2} \to \boxed{V_r \sim \frac{g}{\nu}\frac{V^3}{r_o^7}}$

$$\frac{O(\rho \frac{\partial \bar{v}}{\partial t})}{O(\mu \nabla^2 \bar{v})} \sim \frac{O(\rho \bar{v}\cdot\nabla\bar{v})}{O(\mu\nabla^2\bar{v})} \sim \frac{V_r h}{\nu}\frac{h}{r_f} \sim \boxed{\frac{g V^5}{\nu^2 r_o^{12}} << 1}$$

2)
$$0 = -\frac{\partial P}{\partial z} - \rho g \to P - P_a = \rho g (h-z) \to \frac{\partial P}{\partial r} = \rho g \frac{\partial h}{\partial r}$$

$$\left.\begin{array}{l} 0 = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 V_r}{\partial z^2},\ V_r = 0 \text{ AT } z=0 \\ \frac{\partial V_r}{\partial z} = 0 \text{ AT } z=h \end{array}\right\} \to V_r = -\frac{g h^2}{2\nu}\frac{\partial h}{\partial r}\frac{z}{h}\left(2-\frac{z}{h}\right),\quad \int_0^h V_r dz = -\frac{g h^3}{3\nu}\frac{\partial h}{\partial r}$$

$$\int_0^h \left(\frac{1}{r}\frac{\partial (r V_r)}{\partial r} + \frac{\partial V_z}{\partial z}\right) dz = 0 \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r \int_0^h V_z dz\right) - \frac{\partial h}{\partial r} V_z\Big|_{z=h} + V_z\Big|_{z=h} = 0$$

THE INTERFACE $z - h(r,t) = 0$ IS A FLUID SURFACE $\to -\frac{\partial h}{\partial t} - V_r \frac{\partial h}{\partial r} + V_z = 0$ AT $z = h$

$t=0: h = h_0(r)$

$$\boxed{\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r}\right)}$$

$r=0: \frac{\partial h}{\partial r} = 0$

$r = r_f(t): h = 0$

THE SOLUTION SATISFIES

$$\int_0^{r_f} r \frac{\partial h}{\partial t} dr = 0 \Rightarrow \boxed{\int_0^{r_f} 2\pi h r dr = V}$$

3) $\bar{r} = r/r_o$, $\bar{h} = \frac{h}{V/r_o^2}$, $\tau = \frac{g V^3 t}{\nu r_o^8} \Rightarrow \frac{\partial \bar{h}}{\partial \tau} = \frac{1}{3}\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\bar{h}^3\frac{\partial \bar{h}}{\partial \bar{r}}\right)$

$\tau = 0: \bar{h} = \bar{h}_o(\bar{r})$, $\left(\int_0^1 2\pi \bar{h}_o \bar{r} d\bar{r} = 1\right)$

$\tau > 0: \frac{\partial \bar{h}}{\partial \bar{r}} = 0$ AT $\bar{r} = 0$

$\bar{h} = 0$ AT $\bar{r} = \bar{r}_f(\tau)$ $\left(\int_0^{r_f} 2\pi \bar{h}\bar{r} d\bar{r} = 1 \text{ AT ALL TIMES}\right)$

4) $\frac{h_\infty}{t} \sim \frac{g}{\nu}\frac{h_\infty^4}{r_\infty^2}$, $h_\infty r_\infty^2 \sim V$

$$h_\infty = \left(\frac{\nu V}{g t}\right)^{1/4},\quad r_\infty = \left(\frac{g t V^3}{\nu}\right)^{1/8}$$

5)
$$\frac{d}{d\eta}\left(\eta \frac{H^3}{3}\frac{dH}{d\eta} + \frac{\eta^2 H}{8}\right) = 0 \qquad \begin{array}{l}\eta = 0: \frac{dH}{d\eta} = 0 \\ \eta = \eta_f: H = 0\end{array}$$

$$\Rightarrow H = \left(\frac{9}{16}\right)^{1/3}(\eta_f^2 - \eta^2)^{1/3} \to \boxed{2\pi\int_0^{\eta_f}\left(\frac{9}{16}\right)^{1/3}(\eta_f^2-\eta^2)^{1/3}\eta\, d\eta = 1}$$

$$\eta_f^8 = \frac{r_f^8}{g t V^3/\nu} = \frac{16}{9}\left(\frac{4}{3\pi}\right)^3$$