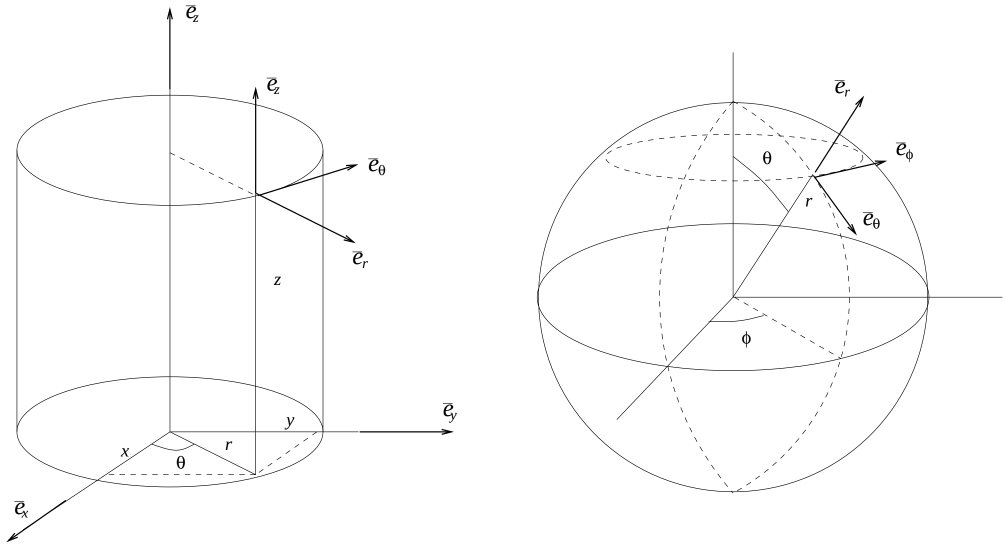

Figure 1: Cartesian, cylindrical, and spherical coordinate systems.

The differential line elements associated with each of the three coordinate systems considered are, respectively, $\mathrm{d}\bar{x} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$, $\mathrm{d}\bar{x} = (\mathrm{d}r, r\mathrm{d}\theta, \mathrm{d}z)$ and $\mathrm{d}\bar{x} = (\mathrm{d}r, r\mathrm{d}\theta, r\sin\theta\mathrm{d}\phi)$. The coefficients that premultiply each of the differentials in the above expressions are the so-called scale factors (h_1, h_2, h_3), which reduce to $(h_1 = 1, h_2 = 1, h_3 = 1)$ for cartesian coordinates, to $(h_1 = 1, h_2 = r, h_3 = 1)$ for cylindrical coordinates, and to $(h_1 = 1, h_2 = r, h_3 = r\sin\theta)$ for spherical coordinates.

The gradient of a scalar function Φ is in general given by

$$\nabla\Phi = \left(\frac{1}{h_1}\frac{\partial\Phi}{\partial x_1}, \frac{1}{h_2}\frac{\partial\Phi}{\partial x_2}, \frac{1}{h_3}\frac{\partial\Phi}{\partial x_3}\right), \tag{1}$$

while its laplacian can be expressed in the form

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial x_3}\right)\right]. \tag{2}$$

Similarly, the divergence of a vector function $\mathbf{a} = a_1\bar{e}_1 + a_2\bar{e}_2 + a_3\bar{e}_3$ is

$$\nabla\cdot\mathbf{a} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3a_1) + \frac{\partial}{\partial x_2}(h_1h_3a_2) + \frac{\partial}{\partial x_3}(h_1h_2a_3)\right], \tag{3}$$

while its curl is given by

$$\begin{aligned}\nabla\wedge\mathbf{a} = \tfrac{1}{h_1h_2h_3}\begin{vmatrix} h_1\bar{e}_1 & h_2\bar{e}_2 & h_3\bar{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix} &= \tfrac{1}{h_2h_3}\left[\tfrac{\partial}{\partial x_2}(h_3a_3) - \tfrac{\partial}{\partial x_3}(h_2a_2)\right]\bar{e}_1 + \\ \tfrac{1}{h_1h_3}\left[\tfrac{\partial}{\partial x_3}(h_1a_1) - \tfrac{\partial}{\partial x_1}(h_3a_3)\right]\bar{e}_2 &+ \tfrac{1}{h_1h_2}\left[\tfrac{\partial}{\partial x_1}(h_2a_2) - \tfrac{\partial}{\partial x_2}(h_1a_1)\right]\bar{e}_3.\end{aligned} \tag{4}$$

Also, the gradient of a vector function produces a tensor according to

$$(\nabla\mathbf{a})_{ii} = \frac{1}{h_i}\frac{\partial a_i}{\partial x_i} + \sum_{k\neq i}\frac{a_k}{h_ih_k}\frac{\partial h_i}{\partial x_k} \quad \text{and} \quad (\nabla\mathbf{a})_{ij} = \frac{1}{h_i}\frac{\partial a_j}{\partial x_i} - \frac{a_i}{h_ih_j}\frac{\partial h_i}{\partial x_j} \quad \text{if} \quad i\neq j, \tag{5}$$

while the divergence of a tensor $\nabla\cdot\bar{\bar{A}}$ gives a vector function of components

$$(\nabla\cdot\bar{\bar{A}})_i = \frac{h_i}{h}\sum_j\frac{\partial}{\partial x_j}\left(\frac{hA_{ij}}{h_ih_j}\right) + \sum_j\frac{A_{ij}+A_{ji}}{h_ih_j}\frac{\partial h_i}{\partial x_j} - \sum_j\frac{A_{jj}}{h_ih_j}\frac{\partial h_j}{\partial x_i}, \tag{6}$$

where $h = h_1h_2h_3$.

For a constant-density fluid, the viscous stress tensor takes the form

$$\bar{\bar{\tau}}' = \mu[(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T] \tag{7}$$

Cartesian coordinates

$$\tau'_{xx} = 2\mu\frac{\partial v_x}{\partial x} \qquad \tau'_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

$$\tau'_{yy} = 2\mu\frac{\partial v_y}{\partial y} \qquad \tau'_{yz} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)$$

$$\tau'_{zz} = 2\mu\frac{\partial v_z}{\partial z} \qquad \tau'_{zx} = \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$$

Cylindrical coordinates

$$\tau'_{rr} = 2\mu\frac{\partial v_r}{\partial r} \qquad \tau'_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau'_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \qquad \tau'_{\theta z} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)$$

$$\tau'_{zz} = 2\mu\frac{\partial v_z}{\partial z} \qquad \tau'_{zr} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

Spherical coordinates

$$\tau'_{rr} = 2\mu\frac{\partial v_r}{\partial r} \qquad \tau'_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau'_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \qquad \tau'_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

$$\tau'_{\phi\phi} = 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right) \qquad \tau'_{\phi r} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

Navier-Stokes equations for ρ and μ constants

$$\nabla \cdot \mathbf{v} = 0, \qquad \rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{f_m}$$

CARTESIAN COORDINATES (x, y, z)

Continuity equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

Momentum equation

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] + \rho f_{m_x}$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] + \rho f_{m_y}$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho f_{m_z}$$

CYLINDRICAL COORDINATES (r, θ, z)

Continuity equation

$$\frac{1}{r} \frac{\partial r v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Momentum equation

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) =$$
$$-\frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] + \rho f_{m_r}$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) =$$
$$-\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \rho f_{m_\theta}$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho f_{m_z}$$

SPHERICAL COORDINATES (r,θ,ϕ)

Continuity equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} = 0$$

Momentum equation

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2} - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right] + \rho f_{m_r}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial\theta} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right] + \rho f_{m_\theta}$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right] + \rho f_{m_\phi}$$