Use the Karman-Pohlhausen method to analyze the boundary layer that develops over a semi-infinite plate \( u_e = U_\infty \). Assume that the velocity profile in the boundary layer has a self-similar form given by

\[
\begin{align*}
\frac{u_e}{U_\infty} &= \sin \left( \frac{\pi y}{2 \delta} \right) \quad \text{for} \quad 0 \leq y \leq \delta \\
\frac{u_e}{U_\infty} &= 1 \quad \text{for} \quad y > \delta.
\end{align*}
\]

Determine the evolution of \( \delta(x) \) and use the result to compute \( \delta^*(x), \theta(x), \) and \( \tau_w(x) \), comparing the approximate solution with the exact results.

\[
\begin{align*}
\delta^* &= \delta \left( 1 - \sin \left( \frac{\pi \delta}{2 \delta} \right) \right) \frac{d}{d \delta} \frac{\delta}{\delta} = \left( 1 - \frac{\pi}{2} \right) \delta = 0.363 \delta \\
\theta &= \delta \left( 1 - \sin \left( \frac{\pi \delta}{2 \delta} \right) \right) \frac{d}{d \delta} \frac{\delta}{\delta} = \left( 1 - \frac{\pi}{2} \right) \delta = 0.1366 \delta \\
\tau_w &= \frac{\gamma}{2} \frac{M u_\infty}{\delta} \approx 1.57 \frac{M u_\infty}{\delta}
\end{align*}
\]

\[
\frac{d}{dx} \left( U_e^2 \theta \right) + U_e \frac{d U_e}{dx} \delta^* = \frac{\tau_w}{\delta} \Rightarrow \left( \frac{\pi}{2} - \frac{1}{2} \right) \frac{U_e}{\delta} \frac{d \delta}{dx} = \frac{\pi}{2} \frac{U_e}{\delta}, \quad x=0: \delta = 0
\]

\[
\delta = \sqrt{\frac{\pi}{4 \gamma} \tau_w \left( \frac{u_e}{U_\infty} \right)^{\frac{1}{2}}} \approx 4.795 \left( \frac{u_e}{U_\infty} \right)^{\frac{1}{2}}
\]

\[
\delta^* = 1.740 \sqrt{\frac{u_e}{U_\infty}} \quad \text{(Blasius 1.721)}
\]

\[
\theta = 0.655 \sqrt{\frac{u_e}{U_\infty}} \quad \text{(Blasius 0.664)}
\]

\[
\tau_w = 0.327 \frac{M u_\infty}{U_\infty} \sqrt{\frac{U_\infty}{u_e}} \quad \text{(Blasius 0.332)}
\]
A thin splitter plate of length $L$ is placed normal to a wall at a location where there exists a stagnation point, so that the velocity for the inviscid flow is given in the first approximation by the potential solution, including an external velocity distribution $U_e = A(L - x)$ over the splitter plate. Use the Karman-Polhaussen method to investigate the boundary layer that develops over the plate. Assume that the velocity profile is given by the distribution $u/U_e = 2y/\delta - (y/\delta)^2$ for $0 \leq y < \delta$ and $u/U_e = 1$ for $y > \delta$. Determine $\delta^*, \theta$, and $\tau_w$ as a function of $\delta(x)$ and use the momentum integral equation to determine $\delta(x)$.

\[
\begin{align*}
\delta^* &= \int_0^\infty (1 - \frac{y}{U_e}) \, dy = \frac{\delta}{3} \\
\theta &= \int_0^\infty (1 - \frac{y}{U_e}) \frac{u}{U_e} \, dy = \frac{2}{15} \delta \\
\tau_w &= \frac{m}{\delta} \left. \frac{du}{dy} \right|_{y=0} = \frac{2m}{\delta} \frac{U_e}{\delta}
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dx} \left( \frac{U_e^2 \theta}{\delta} \right) + U_e \frac{dU_e}{dx} } \frac{\delta^*}{\delta} &= \gamma \frac{U_e^2}{\delta} \frac{d\theta}{dx} + U_e \frac{dU_e}{dx} \left( 2\theta + \delta^* \right) = \frac{2\tau_w}{\delta} \\
\frac{2}{15} A^2 (L - x)^2 \frac{d\delta}{dx} - A^2 (L - x) \frac{3}{5} \delta = 2U_e \frac{A(L-x)}{\delta}
\end{align*}
\]

\[
\begin{align*}
(L-x) \frac{d}{dx} \left( \frac{\delta^2}{U_e^2} \right) - q \left( \frac{\delta^2}{U_e^2} \right) &= 30, \quad \delta(0) = 0 \quad \Rightarrow \quad \frac{\delta}{(U_e^2)}^{1/2} = \left[ \frac{10 (1 - \frac{x}{L})^2}{15 (1 - \frac{x}{L})^3} - 1 \right]^{1/2}
\end{align*}
\]
BL.7) The figure shows schematically the structure of the laminar flow found at the entrance of a plane channel of semi-width \( h \) when the flow Reynolds number is moderately large. Near the entrance the flow is inviscid in the first approximation and the velocity profile is given by the uniform distribution \( v_x = v_o \). The effect of viscosity is initially confined to near-wall boundary layers, whose thickness \( \delta \) increases downstream, as sketched in the figure. As the boundary layers develop, continuity dictates that the velocity deficit near the wall must be compensated by an acceleration of the central inviscid core, whose uniform velocity \( v_x(x) > v_o \) increases downstream as driven by a self-induced pressure gradient. The boundary layers thicken to occupy the whole channel at a distance \( l_e \sim v_o h^2 / \nu \), so that for \( x \gg l_e \) the flow is dominated by viscosity and the resulting velocity profile correspondingly reduces to the parabolic Poiseuille distribution \( v_x = \frac{3}{2} v_o [2(y/h) - (y/h)^2] \).

To obtain approximately the value of \( l_e \), one may investigate the development of the boundary layer in the lower half of the channel and assume that the velocity profile at intermediate distances \( 0 < x < l_e \) includes a boundary-layer region for \( 0 \leq y \leq \delta \) where \( v_x/v_o = 2(y/\delta) - (y/\delta)^2 \) and an inviscid core region for \( \delta \leq y \leq h \) where \( v_x = v_e \).

1. For the presumed boundary-layer profile, obtain the values of \( \delta \), \( \theta \), and \( \tau_w \) as a function of \( \delta(x) \) and \( v_x(x) \).

2. Use continuity to derive an equation giving \( v_x(x) \) as a function of \( \delta(x) \), verifying that when \( \delta = h \) the equation correctly yields \( v_x = \frac{3}{2} v_o \).

3. Write an equation for the auto-induced pressure gradient \( dP/dx \) as a function of \( \delta(x) \).

4. Use Von Karman boundary-layer integral equation to derive a differential equation for the evolution of \( \delta(x) \).

5. Obtain the value of entrance length, giving the result in the form \( l_e / (v_o h^2 / \nu) \) as a quadrature.