Chapter 8

Introduction to Potential Flow

The concept of vorticity

$$\bar{\omega} = \nabla \wedge \bar{v} \tag{8.1}$$

was introduced previously when analyzing the kinematics of fluid motion. Physically, it represents twice the angular velocity with which the fluid particle rotates as a solid body. The present chapter deals with irrotational (or potential) flows, for which the vorticity is zero at every point in the flow field. For simplicity in the presentation, attention will be restricted to incompressible flow, which applies to the motion of liquids and also of gas at low Mach numbers. Complex variable will be seen to facilitate the description of plane flows. The analysis of motion around a circular cylinder will allow us to introduce some aerodynamic concepts including the D-Alembert's paradox and the Kutta-Joukowski formula.

The vorticity equation

A conservation equation for the vorticity can be derived from the momentum equation

$$\frac{\partial \bar{v}}{\partial t} + \nabla(|\bar{v}|^2/2) - \bar{v} \wedge (\nabla \wedge \bar{v}) = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \bar{\bar{\tau}}' + \bar{f}_m. \tag{8.2}$$

For simplicity, attention will be restricted to incompressible flows with conservative mass forces and constant viscosity, for which the above equation takes the form

$$\frac{\partial \bar{v}}{\partial t} + \nabla(|\bar{v}|^2/2) - \bar{v} \wedge \bar{\omega} = -\nabla\left(\frac{p}{\rho} + U\right) + \nu \nabla^2 \bar{v}. \tag{8.3}$$

Since the curl of a gradient is identically zero, taking the curl of this last equation leads to

$$\frac{D\bar{\omega}}{Dt} = \frac{\partial \bar{\omega}}{\partial t} + \bar{v} \cdot \nabla \bar{\omega} = \omega \cdot \nabla \bar{v} + \nu \nabla^2 \bar{\omega}. \tag{8.4}$$

In writing the term $\nabla \wedge (\bar{v} \wedge \bar{\omega})$ we have used the vector identity

$$\nabla \wedge (\bar{v} \wedge \bar{\omega}) = \bar{v}(\nabla \cdot \bar{\omega}) - \bar{\omega}(\nabla \cdot \bar{v}) - \bar{v} \cdot \nabla \bar{\omega} + \bar{\omega} \cdot \nabla \bar{v} \tag{8.5}$$

where the first two terms vanish because of the additional condition that the divergence of a curl is identically zero (i.e., $\nabla \cdot \bar{\omega} = 0$) and the fact that, for an incompressible fluid, the velocity field is solenoidal, that is, $\nabla \cdot \bar{v} = 0$. Equation (8.4) indicates that for an incompressible fluid the variation of the vorticity following a given fluid particle is due to two different contributions. The term $\nu \nabla^2 \bar{\omega}$ represents the diffusion of vorticity due to molecular transport effects (remember that molecular transport was seen to be responsible for the appearance of viscous forces in the momentum equation and of heat conduction in the energy equation). For a given flow with characteristic length and velocity scales L_c and v_c, the relative importance of convective and diffusive transport of vorticity is measured by the Reynolds number, according to

$$\frac{O(\bar{v} \cdot \nabla \bar{\omega})}{O(\nu \nabla^2 \bar{\omega})} \sim \frac{v_c L_c}{\nu}, \tag{8.6}$$

so that vorticity diffusion is negligible for large values of the Reynolds number $v_c L_c / \nu \gg 1$. The term $\bar{\omega} \cdot \nabla \bar{v}$, on the other hand, represents the production or destruction of vorticity by stretching or turning the vortex lines (lines that are tangent to the vorticity vector $\bar{\omega}$ at each point). In planar flows, for which $\bar{\omega}$ is perpendicular to the plane of motion, this term can be seen to vanish. It is of interest that pressure and gravity forces do not appear in (8.4), the physical reason being that these forces act though the center of mass of the fluid particle and therefore cannot change its angular velocity.

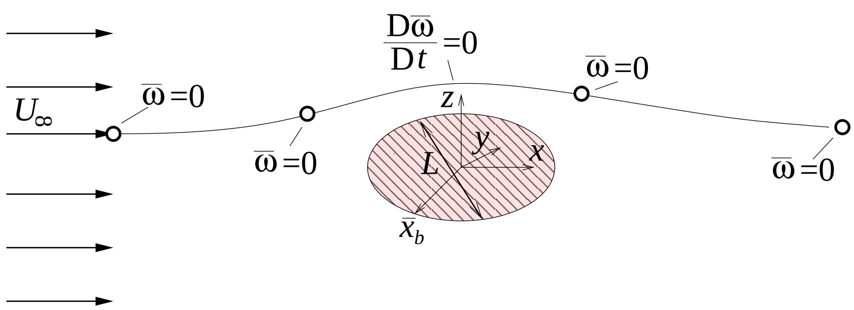

Figure 8.1: Vorticity following a fluid particle in ideal flow about a general three-dimensional body.

Equation (8.4) determines the vorticity of a given incompressible flow. It can be used in particular to demonstrate that the vorticity is zero in some problems of practical interest, an example being the ideal flow about a body, the central problem of interest in aerodynamics. Consider a uniform stream of velocity $\bar{v} = U_\infty \bar{e}_x$ flowing about a body of characteristic size L, as sketched in Fig. 8.1. The evolution of the vorticity of a given fluid particle is determined by (8.4). If the Reynolds number $U_\infty L/\nu$ is large, the diffusion of vorticity is negligible, and the equation reduces to

$$\frac{D\bar{\omega}}{Dt} = \omega \cdot \nabla \bar{v}. \tag{8.7}$$

Since the free-stream vorticity is zero, because $\nabla \wedge (U_\infty \bar{e}_x) = 0$, the evolution of the vorticity for a fluid particle that enters the flow field from the left boundary is obtained by integration of (8.7) with initial condition $\bar{\omega} = 0$, giving $\bar{\omega} = 0$ at any given instant of time. Since that is true for any fluid particle, it is clear that the resulting flow is irrotational, i.e.,

$$\bar{\omega} = \nabla \wedge \bar{v} = 0. \tag{8.8}$$

In assessing the validity of the previous equation, one should note that at large Reynolds numbers vorticity appears confined in a thin boundary layer around the body where diffusion of vorticity from the wall, where it is created due to viscous effects, is significant. If the boundary layer remains attached to the body, as occurs in aerodynamic streamlined bodies, then (8.8) holds everywhere in the flow field outside the boundary layer. Often, however, the boundary layer over blunt bodies separates at a given location, ejecting vorticity into the flow field in the form of a concentrated vortex layer (also called mixing layer) bounding a wake, which is unsteady in many practical situations. The presence of concentrated vorticity invalidates the existence of uniformly-valid potential flow solutions whenever boundary-layer separation occurs.

Potential flow

A flow (viscous or inviscid, compressible or incompressible, steady or unsteady) is said to be potential or irrotational when the vorticity is identically zero at every point in the flow field. This condition (8.8) simplifies the flow description significantly, because it is known from vector analysis that an irrotational vector field can be expressed as a gradient of a scalar potential function. In connection with irrotational flows, we therefore introduce the velocity potential $\phi(\bar{x}, t)$ such that, if (8.8) holds everywhere in the flow field, then the velocity is given by

$$\bar{v} = \nabla\phi, \tag{8.9}$$

where ϕ may have an arbitrary function of time added without a change in $\bar{v}$.
The velocity potential can be used for the description of compressible and incompressible irrotational fluid motion. For the case of incompressible fluid motion treated here, the associated conservation equation to be satisfied by the velocity potential is particularly simple, in that substitution of (8.9) into the corresponding continuity equation

$$\nabla \cdot \bar{v} = 0 \tag{8.10}$$

yields the well-known Laplace equation

$$\nabla^2\phi = 0. \tag{8.11}$$

Once the potential is obtained by integration of (8.11), the associated pressure field can be determined from (8.3) written with $\bar{\omega} = 0$ [1] in the form

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + U\right) = 0, \tag{8.12}$$

which can be integrated to give the generalized Bernoulli's equation

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + U = F(t), \tag{8.13}$$

where the function $F(t)$ is to be determined from boundary information.

[1] For incompressible flow with constant viscosity, $\nabla\cdot\bar{\bar{\tau}}' = \mu\nabla^2\bar{v} = -\mu\nabla\wedge(\nabla\wedge\bar{v})$, so that viscous forces are identically zero if the flow is irrotational.

As an illustrative example, let us consider the incompressible aerodynamic problem sketched in Fig. 8.1. Since the flow is irrotational, the velocity field is given by (8.9), where the velocity potential is determined by integration of (8.11) with boundary conditions

$$\begin{cases} |\bar{x}| \gg L: & \phi = U_\infty x \\ \bar{x} = \bar{x}_b: & \bar{n} \cdot \nabla\phi = 0 \end{cases} \tag{8.14}$$

corresponding to the free-stream velocity $\bar{v} = U_\infty \bar{e}_x$ far from the body and the condition of zero normal velocity $\bar{v} \cdot \bar{n} = 0$ on the body surface $\bar{x} = \bar{x}_b$. The corresponding pressure field is given by (8.13), yielding for instance

$$\frac{p}{\rho} + g\mathcal{Z} + \frac{1}{2}\nabla\phi \cdot \nabla\phi = \frac{P_\infty}{\rho} + \frac{1}{2}U_\infty^2 \tag{8.15}$$

for steady motion with gravity forces included, with $P_\infty = (p + \rho g \mathcal{Z})_\infty$ being the value of the reduced pressure far from the body.

The description given by (8.11) and (8.13) applies in general to irrotational flows of perfect liquids, but also to other fluids, as long as they behave as effectively incompressible for the flow conditions considered. That would be the case of ideal air flow about a body when the Mach number of the approaching free stream U_∞/a_∞ is sufficiently small. To see it, note that the condition of large Reynolds number $U_\infty L/\nu \gg 1$ implies that the flow is isentropic. Integration of

$$\frac{\mathrm{D}s}{\mathrm{D}t} = c_v \frac{\mathrm{D}}{\mathrm{D}t}\left[\ln(p/\rho^\gamma)\right] = 0 \tag{8.16}$$

for a given fluid particle yields

$$\frac{p}{\rho^\gamma} = \frac{p_\infty}{\rho_\infty^\gamma} \tag{8.17}$$

if the conditions are uniform in the free stream. On the other hand, for large Reynolds numbers, the spatial pressure differences Δp can be estimated from the balance $\rho\bar{v} \cdot \nabla\bar{v} \sim -\nabla p$ to give

$$\Delta p/p_\infty \sim \rho_\infty U_\infty^2/p_\infty \sim \gamma(U_\infty/a_\infty)^2. \tag{8.18}$$

Therefore, for air flows where the free-stream velocity is much smaller than the velocity of sound (i.e., low-Mach-number flows), the relative spatial pressure differences are of order $(U_\infty/a_\infty)^2 \ll 1$, giving also negligibly small density differences $\Delta\rho/\rho_\infty \sim (U_\infty/a_\infty)^2 \ll 1$, according to (8.17). Under those conditions, the irrotational velocity and pressure fields of the air flow are determined in the first approximation by the incompressible equations (8.11) and (8.13), with errors, due to compressibility, that would be of order $(U_\infty/a_\infty)^2 \ll 1$.

Planar potential flows

Incompressible potential flows, determined by (8.11) and (8.13), are relatively easy to analyze, because there is a vast amount of mathematical information about Laplace equation and its solutions. A particularly simple situation arises when the flow field is planar, with a velocity that can be expressed in cartesian coordinates according to

$$\bar{v} = v_x\bar{e}_x + v_y\bar{e}_y. \tag{8.19}$$

The condition of irrotational motion (8.8) leads to

$$\omega_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0 \tag{8.20}$$

while continuity provides

$$\nabla \cdot \bar{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{8.21}$$

Because the flow is irrotational, there exists a velocity potential such that

$$v_x = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v_y = \frac{\partial \phi}{\partial y} \tag{8.22}$$

which, since the flow is incompressible, satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{8.23}$$

obtained by substituting (8.22) into (8.21). On the other hand, for a planar incompressible flow we can define a stream function ψ such that [2]

$$v_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v_y = -\frac{\partial \psi}{\partial x}, \tag{8.24}$$

which, since the flow is irrotational, satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{8.25}$$

obtained by substituting (8.24) into (8.20).
The flow can be analyzed by introducing the so-called complex potential

$$f(z) = \phi(x,y) + \imath\psi(x,y), \tag{8.26}$$

defined to be an *analytic function* [3] in the complex plane $z = x + \imath y = re^{\imath\theta}$. The definition suits perfectly our needs, because if $f(z)$ is analytic, then its real and imaginary parts satisfy Laplace equation (i.e., equations (8.23) and (8.25)) along with the Cauchy-Riemann conditions

$$v_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{8.29}$$

[2]The definition and properties of the stream function are found in Chapter 2.

[3]A function of complex variable

$$f(z) = \mathcal{R}(x,y) + \imath\mathcal{I}(x,y) \tag{8.27}$$

is said to be analytic in a region if it is differentiable at every point of that region, that is, at every point the partial derivatives $\partial\mathcal{R}/\partial x$, $\partial\mathcal{R}/\partial y$, $\partial\mathcal{I}/\partial x$, and $\partial\mathcal{I}/\partial y$, are continuous and satisfy the so-called Cauchy-Riemann equations

$$\frac{\partial \mathcal{R}}{\partial x} = \frac{\partial \mathcal{I}}{\partial y} \qquad \text{and} \qquad \frac{\partial \mathcal{R}}{\partial y} = -\frac{\partial \mathcal{I}}{\partial x}, \tag{8.28}$$

which guarantee that the value of $\mathrm{d}f/\mathrm{d}z$ is independent of the direction in which it is calculated. If a function is analytic, then it can be demonstrated that its real and imaginary parts satisfy Laplace equation.

In other words, for every analytic function $f(z)$ the real part is automatically a valid velocity potential and the imaginary part is a valid stream function. The derivative of the complex potential is the so-called complex velocity

$$\frac{\mathrm{d}f}{\mathrm{d}z} = v_x - \imath v_y = (v_r - \imath v_\theta)e^{-\imath\theta}, \tag{8.30}$$

as can be seen by straightforward derivation of (8.27) (for instance, in the direction $z = x$ or $z = \imath y$).

An approach to be followed below in investigating planar potential flows involves consideration of simple analytic functions that represent elementary flows of interest, such as sources, sinks, or vortices. More complex flows can be constructed by superposition of these elementary solutions, a procedure that is always admissible because the Laplace equation is linear. This approach, albeit indirect, provides interesting results for some flows of interest and avoids difficulties of solving partial differential equations. Note that, while the velocity field obtained by adding two different complex potentials is the sum of the associated velocity fields, the pressure is not, because the corresponding equation (8.13) is nonlinear. Below, we proceed to introduce some elementary complex potentials of interest and use them subsequently for the analysis of flow over a circular cylinder.

Simple analytic functions

The simplest analytic function is proportional to z, $f(z) = cz$, with a proportionality constant that can be selected to give

$$f(z) = U_\infty e^{-\imath\alpha} z. \tag{8.31}$$

The above complex potential represents a **uniform stream** of velocity U_∞ inclined at an angle α to the x axis, as shown in Fig. 8.2. Taking the derivative of (8.31) to compute the complex velocity (8.30) yields

$$\frac{\mathrm{d}f}{\mathrm{d}z} = U_\infty \cos\alpha - \imath U_\infty \sin\alpha, \tag{8.32}$$

so that

$$v_x = U_\infty \cos\alpha \quad \text{and} \quad v_y = U_\infty \sin\alpha. \tag{8.33}$$

On the other hand, the velocity potential and the stream function for the flow are

$$\phi = U_\infty(x\cos\alpha + y\sin\alpha) \quad \text{and} \quad \psi = U_\infty(y\cos\alpha - x\sin\alpha), \tag{8.34}$$

as obtained from taking the real and imaginary parts of $f(z)$, with the latter yielding $y = x\tan\alpha + \textit{constant}$ for the equations of the streamlines (straight lines with inclination α with respect to the horizontal line).

The complex potential for a **source** of strength Q centered at the origin is given by

$$f(z) = \frac{Q}{2\pi}\ln(z), \tag{8.35}$$

with $\ln(z) = \ln(r) + \imath\theta$ taken as the principal part corresponding to $0 < \theta < 2\pi$. Taking the derivative of (8.35) yields

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{Q}{2\pi z} = \frac{Q}{2\pi r}e^{-\imath\theta} \tag{8.36}$$

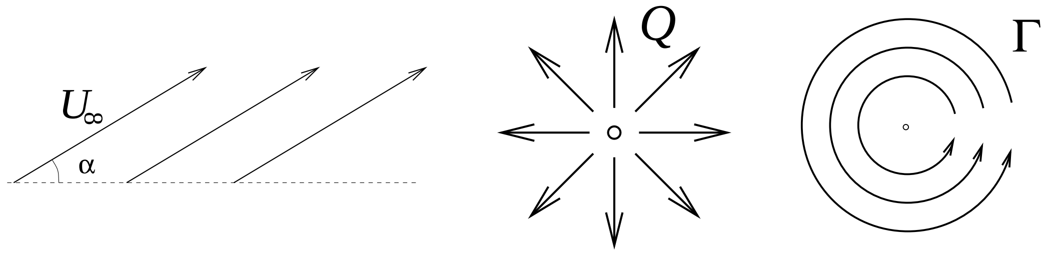

Figure 8.2: Uniform flow, source, and vortex elementary solutions.

for the complex velocity, so that

$$v_r = \frac{Q}{2\pi r} \quad \text{and} \quad v_\theta = 0, \tag{8.37}$$

as follows from comparison with (8.30). The volume flux (per unit length perpendicular to the plane of motion) across a circle of radius r is given by $\int_0^{2\pi} v_r r \mathrm{d}\theta = Q$, equal to the strength of the source. It is straightforward to generalize (8.35) for a source centered at $z = z_o$ by writing

$$f(z) = \frac{Q}{2\pi} \ln(z - z_o), \tag{8.38}$$

with Q replaced with $-Q$ to represent a sink.
The complex potential for a **vortex** of strength Γ centered at the origin is given by

$$f(z) = -\frac{\Gamma \imath}{2\pi} \ln(z), \tag{8.39}$$

with corresponding velocity components

$$v_r = 0 \quad \text{and} \quad v_\theta = \frac{\Gamma}{2\pi r}, \tag{8.40}$$

obtained by derivation. The positive (counterclockwise) circulation associated with this flow can be computed to be $\int_0^{2\pi} v_\theta r \mathrm{d}\theta = \Gamma$, equal to the strength of the vortex. Note that for vortices with clockwise (negative) circulation, associated with lifting bodies, one should use a plus sign on the right-hand-side of (8.39). A general vortex of circulation Γ centered at $z = z_o$ is in general given by

$$f(z) = -\frac{\Gamma \imath}{2\pi} \ln(z - z_o), \tag{8.41}$$

Another elementary solution of interest is the **doublet**. It can be seen as the limiting solution for the flow induced by a source and a sink of equal strenghts Q located at an infinitesimally small distance $2a$

$$f(z) = \frac{Q}{2\pi} \ln(z + a) - \frac{Q}{2\pi} \ln(z - a) = \frac{Q}{2\pi} [\ln(1 + a/z) - \ln(1 - a/z)]. \tag{8.42}$$

The solution at distances $z \gg a$ is given in the first approximation by

$$f(z) = \frac{M}{z}, \tag{8.43}$$

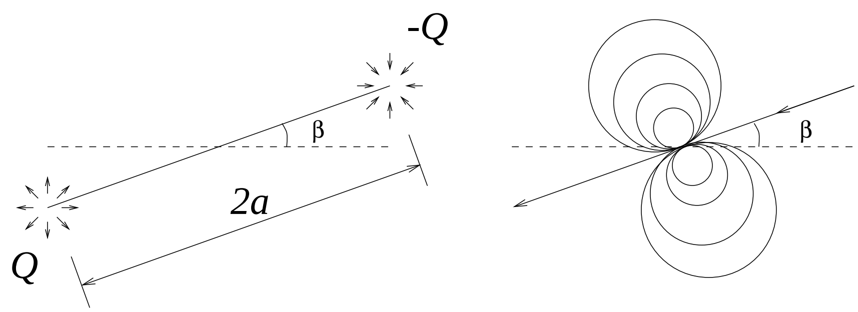

Figure 8.3: The doublet as limiting superposition of a source and a sink.

where $M = Qa/\pi$ is the strenght of the doublet. It can be seen that the corresponding streamlines are circles of radius R centered at $z = \pm R\imath$. The solution can be generalized to doublets centered at $z = z_o$ with inclination β according to

$$f(z) = \frac{Me^{\imath\beta}}{z}. \tag{8.44}$$

The reader can verify that the doublet can be also obtained as limiting solution for two close vortices with opposite circulation.
The above elementary analytic functions, summarized in the table below, can be combined linearly to generate more complex flows by superposition. For instance, the complex potential for the flow over a Rankine oval can be obtained by superposing a uniform stream with a source located at $z = -a$ and a sink of equal strength located at $z = a$, yielding $f(z) = U_\infty z + Q\ln(z+a)/(2\pi) - Q\ln(z-a)/(2\pi)$.

Elementary flow	Complex potential $f(z)$
Uniform flow with inclination α	$U_\infty e^{-\imath\alpha} z$
Source with volume rate Q	$Q\ln(z - z_o)/(2\pi)$
Vortex with circulation Γ	$-\imath\Gamma\ln(z - z_o)/(2\pi)$
Doublet of strength $Me^{\imath\beta}$	$Me^{\imath\beta}/(z - z_o)$

Circular cylinder without circulation

Consider the superposition of a uniform flow of velocity U_∞ and a doublet located at the origin with strength $U_\infty a^2$. The associated complex potential

$$f(z) = U_\infty\left(z + \frac{a^2}{z}\right) \tag{8.45}$$

corresponds to the irrotational flow of a uniform stream about a circular cylinder of radius a. To prove it, note that the stream function $\psi = U_\infty y[1 - a^2/(x^2+y^2)]$ obtained as the imaginary part of (8.45) vanishes on the cylinder surface, which is therefore a stream line of the flow, that is, the condition of zero normal velocity $\bar{v}\cdot\bar{n} = 0$ is satisfied, whereas far from the body the velocity approaches the free stream solution $\bar{v} = U_\infty \bar{e}_x$, as can be seen by evaluating the complex velocity

$$\frac{\mathrm{d}f}{\mathrm{d}z} = U_\infty \left(1 - \frac{a^2}{z^2}\right) \tag{8.46}$$

as $z \gg a$. The velocity field, given in cylindrical coordinates by

$$v_r = U_\infty \cos(\theta)\left(1 - \frac{a^2}{r^2}\right) \qquad \text{and} \qquad v_\theta = -U_\infty \sin(\theta)\left(1 + \frac{a^2}{r^2}\right) \tag{8.47}$$

can be used in (8.13) to give the corresponding pressure distribution

$$p - p_\infty = \frac{1}{2}\rho U_\infty^2 \left(1 - \frac{v_r^2 + v_\theta^2}{U_\infty^2}\right) \tag{8.48}$$

everywhere in the flow field.

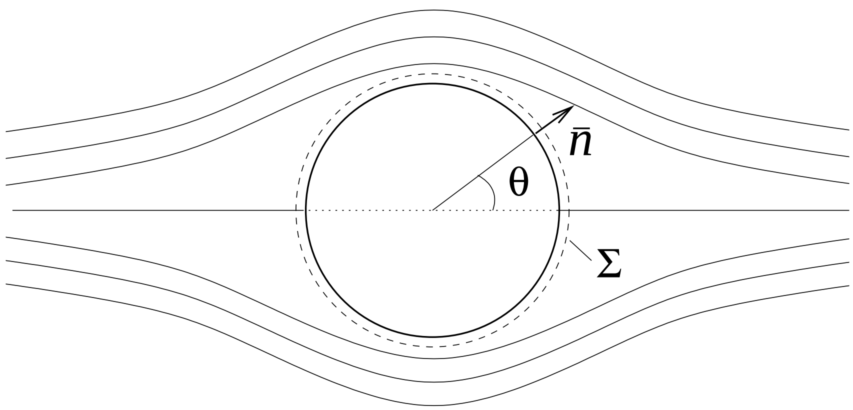

Figure 8.4: Flow about a circular cylinder.

The flow exhibits two stagnation points, located at $z = \pm a$, as can be seen from (8.46). The velocity distribution on the surface of the cylinder is given by

$$v_\theta(\theta) = -2U_\infty \sin(\theta), \tag{8.49}$$

so that the corresponding pressure distribution becomes

$$p - p_\infty = \frac{1}{2}\rho U_\infty^2 \left[1 - 4\sin^2(\theta)\right]. \tag{8.50}$$

For large Reynolds numbers one may neglect viscous forces in the first approximation when computing the force acting on the cylinder $\bar{F} = -\int_\Sigma (p - p_\infty)\bar{n}\mathrm{d}\sigma$. The transverse component $L = -\int_\Sigma (p - p_\infty) n_y \mathrm{d}\sigma$, called lift, is given by

$$L = -\int_0^{2\pi} (p - p_\infty)\sin(\theta) a\mathrm{d}\theta = -\frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} [1 - 4\sin^2(\theta)]\sin(\theta)\mathrm{d}\theta = 0, \tag{8.51}$$

a result that could have been anticipated in view of the symmetry of the problem. Somewhat more surprising is the result to be obtained for the drag force $D = -\int_\Sigma (p - p_\infty) n_x \mathrm{d}\sigma$ (the streamwise component of the force acting on the cylinder), which can be computed according to

$$D = -\int_0^{2\pi} (p - p_\infty)\cos(\theta) a\mathrm{d}\theta = -\frac{1}{2}\rho U_\infty^2 a \int_0^{2\pi} [1 - 4\sin^2(\theta)]\cos(\theta)\mathrm{d}\theta = 0. \tag{8.52}$$

This last result, obtained here for the circular cylinder, is a manifestation of the so-called *D-Alembert's paradox*: the steady motion of solid bodies moving in an incompressible ideal fluid results in zero drag. The resolution of this paradox is due to Prandtl, who introduced the concept of boundary layer, as explained below.

Circular cylinder with circulation

The potential solution given in (8.45) is unique only if one imposes the additional condition that the circulation around the cylinder be zero. In other words, the solution

$$f(z) = U_\infty \left(z + \frac{a^2}{z} \right) + \frac{\imath\Gamma}{2\pi} \ln\left(\frac{z}{a}\right), \tag{8.53}$$

obtained by superposing a vortex of strength $-\Gamma$ located at the center of the cylinder, is also a valid solution, in that $r = a$ is still a stream line and one still recovers for $z \gg a$ the free-stream velocity $\bar{v} = U_\infty \bar{e}_x$. Although it is not necessary in the following, in defining (8.53) we have added an arbitrary constant $\imath\Gamma \ln(a)/(2\pi)$, so that the constant value of ψ on the cylinder surface is still given by $\psi = 0$. The complex velocity

$$\frac{\mathrm{d}f}{\mathrm{d}z} = U_\infty \left(1 - \frac{a^2}{z^2}\right) + \frac{\imath\Gamma}{2\pi z}, \tag{8.54}$$

is modified by the presence of the vortex, yielding

$$\frac{z}{a} = \pm(1 - B^2)^{1/2} - \imath B \tag{8.55}$$

for the location of the stagnation points, where the value of the parameter $B = \Gamma/(4\pi a U_\infty)$ determines the structure of the flow field according to the schematic shown in Fig. 8.5.

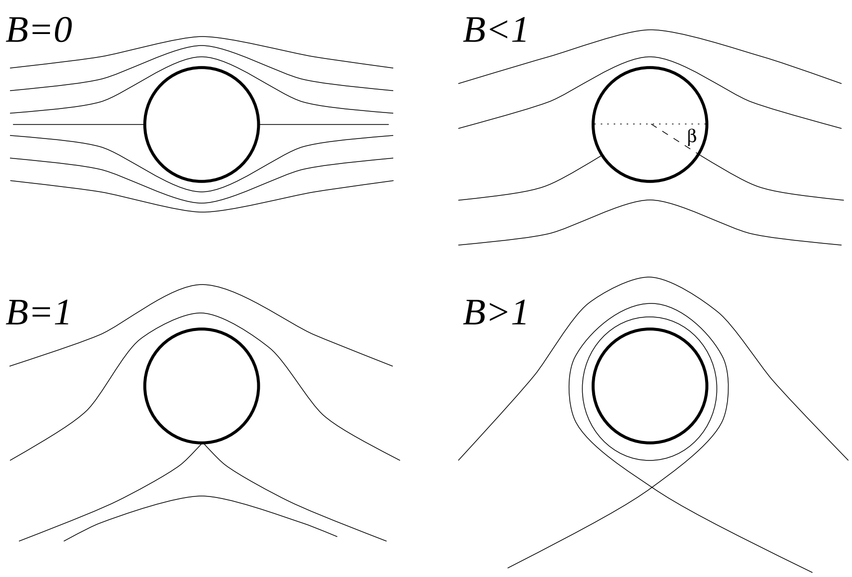

Figure 8.5: Flow about a circular cylinder with circulation.

If $B < 1$ both stagnation points are located on the surface of the cylinder, at $z/a = \pm\cos(\beta) - \imath\sin(\theta)$, where $\sin\beta = B$. For $B = 1$ both points collapse at a single location $z = -\imath a$, whereas for $B > 1$ there appears a single stagnation point outside the cylinder, given by $z/a = -\imath(B + (B^2 - 1)^{1/2})$. The modified velocity and pressure distributions on the cylinder are given by

$$v_\theta(\theta) = -2U_\infty \sin(\theta) - \frac{\Gamma}{2\pi a}, \tag{8.56}$$

and

$$p - p_\infty = \frac{1}{2}\rho U_\infty^2 \left[1 - \left(\frac{\Gamma}{2\pi a U_\infty}\right)^2 - 4\sin^2(\theta) - \frac{2\Gamma\sin(\theta)}{\pi a U_\infty}\right]. \tag{8.57}$$

Note that the pressure still exhibits symmetry about the y axis, i.e., $p(\theta) = p(\pi - \theta)$, so that the resulting force has a zero drag component $D = -\int_0^{2\pi}(p - p_\infty)\cos(\theta) a d\theta = 0$. However, the presence of the vortex destroys the symmetry about the x axis, because it accelerates (decelerates) the flow on the upper (lower) half of the cylinder, reducing (increasing) the pressure there. As a result, a nonzero lift force appears, to be computed from the first equation in (8.51) to give the so-called *Kutta-Joukowski formula*

$$L = \rho U_\infty \Gamma. \tag{8.58}$$

The value of the circulation Γ therefore determines the location of the stagnation points and the lift force acting on the cylinder.