PIV measurements in a wind tunnel have provided the two velocity profiles shown in the figure below.

Profile 1: \[ U = \begin{cases} 3(z - 2/3) & \text{for } 2/3 \leq z \leq 1 \\ 0 & \text{for } 1/3 \leq z \leq 2/3 \\ 3(z - 1/3) & \text{for } 0 \leq z \leq 1/3 \end{cases} \]

Profile 2: \[ U = \begin{cases} 1 & \text{for } 2/3 \leq z \leq 1 \\ 6(z - 1/2) & \text{for } 1/3 \leq z \leq 2/3 \\ -1 & \text{for } 0 \leq z \leq 1/3 \end{cases} \]

Note that both display an inflection point. To investigate their inviscid stability, we can use Rayleigh's equation together with the piecewise linear characteristics, which are an approximate representation for an observer moving with the mean velocity (they are indicated by the red broken lines in the figure). Determine in each case the explicit eigenvalue relation \( c(k) \) and discuss the resulting stability characteristics. Could you have anticipated the result with use made of Fjørtoft's condition?

---

Profile 1:

\[ \Psi = A_1 e^{kx} + B_1 e^{-kx} \]

\[ \hat{V} = A_2 e^{ikx} + B_2 e^{-ikx} \]

\[ A_2 e^{2kx} \left[ 2ck - 3(1 - e^{2kx}) \right] = -B_2 \left[ 2ck - 3 \left( e^{2kx_1} \right) \right] \]

\[ A_2 \left[ 2ck + 3 \left( e^{2kx_1} \right) \right] = -B_2 \left[ 2ck + 3 \left( 1 - e^{2kx_1} \right) \right] \]

Profile 2:

\[ z = 1: \quad A_1 e^{ikx} + B_1 e^{-ikx} = 0 \]

\[ z = \frac{2}{3}: \quad \left\{ \begin{array}{l} -c \left( A_2 e^{2ikx} - B_2 e^{-2ikx} \right) - 3 \left( A_1 e^{ikx} + B_1 e^{-ikx} \right) = -c \left( A_2 e^{ikx} + B_2 e^{-ikx} \right) \\ A_1 e^{2ikx} + B_2 e^{-2ikx} = A_2 e^{ikx} + B_2 e^{-ikx} \end{array} \right. \]

\[ z = \frac{1}{3}: \quad \left\{ \begin{array}{l} -c \left( A_2 e^{ikx} - B_2 e^{-ikx} \right) - 6 \left( A_1 e^{ikx} + B_1 e^{-ikx} \right) = -c \left( A_2 e^{ikx} + B_2 e^{-ikx} \right) \\ A_2 e^{ikx} + B_2 e^{-ikx} = A_2 e^{ikx} + B_2 e^{-ikx} \end{array} \right. \]

\[ z = 0: \quad A_1 + B_1 = 0 \]

\[ A_2 e^{2kx} \left[ (1 - c)k - 3 \left( 1 - e^{2kx} \right) \right] = -B_2 \left[ (1 - c)k - 3 \left( e^{2kx_1} - 1 \right) \right] \]

\[ A_2 \left[ (1 + c)k - 3 \left( e^{2kx_1} - 1 \right) \right] = -B_2 \left[ (1 + c)k - 3 \left( 1 - e^{2kx_1} \right) \right] \]

\[ c^2 = 1 - \frac{q^2 \left( e^{2kx_1} \right)^2}{k^2 \left( e^{2kx_1} \right)^2} \left[ e^{2kx_1} \left( \frac{2kx_1}{k^2} \right) \right] \]

\[ \text{Unstable for } k < k_0 = 4.195 \]

\( (q \kappa)_{x_0} \approx 0.87 \text{ for } k = 2.3 \)
Consider the two-dimensional inviscid stability of a planar jet defined by the basic velocity profile \( U = 1 \) for \( |z| \leq 1 \) and \( U = 0 \) for \( |z| > 1 \). Obtain the explicit eigenvalue relation \( c(k) \), considering separately the sinuous and varicose modes (see exercise 8.10 in Drazin).

\[
\begin{align*}
\text{EVEN } \hat{\psi} \quad \text{(SYMMETRIC OR SINUOUS MODES)} & \quad \hat{\psi}(z) = \hat{\psi}(-z) \\
\text{AT } z > 1: & \quad \hat{\psi} = B_1 e^{-kz} \\
\text{AT } z = 1: & \quad \hat{\psi} = A_2 \left( e^{kz} + e^{-kz} \right) \\
\text{AT } z < 1: & \quad \hat{\psi} = B_1 e^{kz} \\
\text{AT } z = \pm i: & \quad kCB_1 e^{-kz} = (1-c)KA_1 \left( e^k - e^{-k} \right) \\
& \quad \frac{B_1 e^{-k}}{1-c} = A_2 \left( e^k + e^{-k} \right) \\
\frac{C}{k} &= \frac{1 \pm c \sqrt{1 + \coth^2(k)}}{1 + \coth^2(k)} \quad \text{UNSTABLE TO ALL WAVE NUMBERS}
\end{align*}
\]

\[
\begin{align*}
\text{ODD } \hat{\psi} \quad \text{(ANTISYMMETRIC OR VARICOSE MODES)} & \quad \hat{\psi}(z) = -\hat{\psi}(-z) \\
\text{AT } z > 1: & \quad \hat{\psi} = B_1 e^{kz} \\
\text{AT } z = 1: & \quad \hat{\psi} = A_2 \left( e^{-kz} - e^{kz} \right) \\
\text{AT } z < 1: & \quad \hat{\psi} = -B_1 e^{kz} \\
\text{AT } z = \pm i: & \quad kCB_1 e^{kz} = (1-c)KA_1 \left( e^k - e^{-k} \right) \\
& \quad \frac{B_1 e^{k}}{-c} = A_2 \left( e^k - e^{-k} \right) \\
\frac{C}{k} &= \frac{1 \pm c \sqrt{1 - \tanh^2(k)}}{1 - \tanh^2(k)} \quad \text{UNSTABLE TO ALL WAVE NUMBERS}
\end{align*}
\]

\[
\begin{align*}
C_e &= \pm \frac{\sqrt{\coth^2 k}}{1 + \coth^2 k} \quad \text{UNSTABLE TO ALL WAVE NUMBERS} \\
C_e &= \pm \frac{\sqrt{\tanh^2 k}}{1 + \tanh^2 k} \quad \text{ALSO UNSTABLE TO ALL WAVE NUMBERS}
\end{align*}
\]

The varicose and sinuous modes have identical growth rates \( kC \) but different phase velocities \( C_\gamma \).
(Adapted from exercises 8.21 and 8.22 in Drazin) Consider a Boussinesq fluid satisfying the inviscid (dimensional) equations

\[
\nabla \cdot \tilde{V} = 0
\]
\[
\frac{\partial \tilde{V}}{\partial t} + \tilde{V} \cdot \nabla \tilde{V} = -\nabla \left( \frac{p + \rho g z}{\rho_o} \right) + \alpha (T - T_o) g \tilde{e}_z
\]
\[
\frac{\partial T}{\partial t} + \tilde{V} \cdot \nabla T = 0
\]

- Use the characteristic velocity \( U_c \) and characteristic length \( L_c \) to rewrite the equations in the alternative dimensionless form

\[
\nabla \cdot \tilde{v} = 0
\]
\[
\frac{\partial \tilde{v}}{\partial t} + \tilde{v} \cdot \nabla \tilde{v} = -\nabla \pi + \theta \tilde{e}_z
\]
\[
\frac{\partial \theta}{\partial t} + \tilde{v} \cdot \nabla \theta = 0,
\]

defining clearly the dimensionless variables \( \tilde{v}, \pi, \) and \( \theta \).

Consider now the case of stratified parallel flow between two horizontal surfaces \( z = z_1 \) and \( z = z_2 \).

- Show that the basic solution \( \tilde{v} = U(z) \tilde{e}_x \) and \( \theta = \Theta(z) \) satisfies the conservation equations. Determine the associated basic pressure profile \( \pi = \Pi(z) \).

- Introduce the expressions \( v_x = U(z) \tilde{e}_x + v'_x(x, z, t), \; v_z = v'_z(x, z, t), \; \theta = \Theta(z) + \theta'(x, z, t), \) and \( \pi = \Pi(z) + \pi'(x, z, t) \) and linearize the resulting conservation equations for the perturbations \( (v'_x, v'_z, \theta', \pi') \).

- Consider the normal-mode expansion \( (v'_x, v'_z, \theta', \pi') = (\tilde{v}_x, \tilde{v}_z, \tilde{\theta}, \tilde{\pi})e^{ki(x-ct)} \) and show that the eigenvalue problem can be written in the form

\[
(U - c) \left( k^2 \tilde{v}_x - \frac{d^2 \Theta}{dz^2} \right) + \left( \frac{1}{U - c} \frac{d \Theta}{dz} - \frac{d^2 U}{dz^2} \right) \tilde{v}_z = 0; \quad \tilde{v}_z = 0 \text{ at } z = z_1, z_2
\]

This is called the Taylor-Goldstein equation, with \( (d\Theta/dz)^{1/2} \) sometimes called the Brunt-Väisälä frequency, which is expressed here in nondimensional form\(^5\).

- Show that the condition

\[
\frac{d \Theta}{dz} > \frac{1}{4} \left( \frac{dU}{dz} \right)^2
\]

guarantees the stability of the solution, i.e., stratification tends to stabilize the flow when hotter (lighter) fluid rests on top of colder (heavier) fluid.

\(^5\)Note that it is straightforward to extend the Taylor-Goldstein equation to situations in which the density differences are due to changes in composition, rather than changes in temperature, by defining the square of the dimensionless Brunt-Väisälä frequency in the alternative form

\[
N^2 = -\frac{g}{\rho} \frac{d \rho}{dz} / \left( \frac{U_c}{L} \right)^2
\]
\[ \nabla \cdot \vec{v} = 0, \quad \frac{d\vec{v}}{dt} + \nabla \cdot \vec{v} = -\nabla \psi + \frac{\rho}{\rho_0} \frac{d\theta}{dt} + \vec{v} \times \frac{d\vec{v}}{dt} = 0 \]

**FUNDAMENTAL PROPERTIES**

\[ \frac{d}{dx} \left( \frac{\vec{v}}{U_c} \right) = \vec{v} \left( \frac{U_c}{x} \right) \quad \text{and} \quad \theta = \Theta(z) \quad \text{satisfy all EOS.} \]

**LINEARIZED EQUATIONS**

\[ \frac{d^2 \hat{v}_x}{dz^2} + \frac{1}{(U-c)} \frac{d \hat{v}_x}{dz} - \frac{\hat{v}_x}{(U-c)} = \frac{d \Theta}{dz} \]

**NORMAL MODES**

\[ \hat{v}_x(z) = \int_0^z \Theta(z') \left( \frac{U_c}{z} \right) dZ \]

**SOLVING EQUATIONS**

\[ \int_0^z \Theta(z') \left( \frac{U_c}{z} \right) dZ = \hat{v}_x(z) \]

**REDUCED EQUATION**

\[ \frac{d}{dz} \left( \frac{dH}{dz} \right) - \left( \frac{d^2 \hat{v}_x}{dz^2} \right) = 0 \quad \text{in} \quad z \in [z_1, z_2] \]

**IMAGINARY PART**

\[ C_i = \frac{1}{4} \left( \frac{dU}{dz} \right)^2 \quad \text{for} \quad z_1 \leq z \leq z_2 \]

**CONCLUSION**

\[ C_i = 0 \]
(Adapted from exercise 8.14 in Drazin) Consider the stability of an unconfined planar jet described by Bickley's velocity profile \( U = \text{sech}^2(z) \).

- Determine the location of the inflection point \( z = z_s \) and the value of \( U_s = U(z_s) \), showing that the basic velocity profile satisfies Rayleigh-Fjørtoft's conditions for instability.

- Use Tollmien's theoretical arguments (1935) to deduce that the upper bound of the range of unstable wave numbers \( 0 < k < k_N \) corresponds to a neutrally stable solution with \( c_N = \frac{2}{3} \).

- Show that the value of \( k_N \) is obtained by integration of

\[
\frac{d^2 \hat{\psi}}{dz^2} - \left( k_N^2 - 6 \text{sech}^2(z) \right) \hat{\psi} = 0; \quad \hat{\psi}(\pm \infty) = 0.
\]

- Determine the value of \( k_N \) and the corresponding neutral eigenfunction \( \hat{\psi}_N \). Consider separately the solutions for the sinuous and varicose modes.

- Determine by numerical integration the growth rate of the disturbances \( kc_i \) and the phase velocity \( c_r \) in the unstable range of wave numbers \( 0 < k < k_N \), considering separately the sinuous and varicose modes.

- Use your numerical results to compute the vorticity distribution of the perturbed flow (the sum of the base-flow vorticity and the vorticity perturbation)

\[
\omega_y = \frac{dU}{dz} + \epsilon e^{k z} \text{Re} \left[ \left( \frac{d^2 \hat{\psi}}{dz^2} - k^2 \hat{\psi} \right) e^{k (z - c_r \epsilon)} \right]
\]

where \( \epsilon \) is introduced as an arbitrary measure of the perturbation for illustrative purposes. Evaluate the results for the sinuous and varicose modes with the largest growth rate \( k c_i \) (use \( \epsilon = 0.2 \) and \( t = 0 \) when plotting \( \omega_y \)).

\[
U = \text{sech}^2(z), \quad \frac{dU}{dz} = 2 \text{sech}^2(z) \left[ z - 3 \text{sech}^2(z) \right] = 0 \rightarrow \frac{U_s}{z_s} = \frac{2}{3}
\]

\[
\left( \text{sech}^2(z) \right) \left[ \frac{d^2 \hat{\psi}}{dz^2} - k_N^2 \hat{\psi} \right] = 0 \rightarrow \hat{\psi}(\pm \infty) = 0
\]

\[
\hat{\psi} = C_1 P_2^{\text{yn}} \left[ \tanh(z) \right] + C_2 Q_2^{\text{yn}} \left[ \tanh(z) \right]
\]

\[
\hat{\psi}(\pm 0) = 0 \rightarrow C_2 = 0 \quad \text{because } Q_2 \left[ \tanh(z) \right] = 0
\]

\[
\hat{\psi}(\pm 1) = 0 \rightarrow C_1 = \frac{1}{2} \quad \text{because } P_2 \left[ \tanh(z) \right] = \frac{1}{2}
\]

**Varicose Mode**

\[
\hat{\psi}(z = 0) = 0 \rightarrow P_2 \left[ \tanh(z) \right](0) = 0 \rightarrow \hat{\psi}(z = 0) = 0
\]

\[
\hat{\psi}(z = 1) = 0 \rightarrow Q_2 \left[ \tanh(z) \right](1) = 0 \rightarrow \hat{\psi}(z = 1) = 0
\]

**Sinuous Mode**

\[
\frac{d\hat{\psi}}{dz}(z = 0) = 0 \rightarrow Q_2 \left[ \tanh(z) \right](0) = 0
\]

\[
\hat{\psi}(z = 1) = 0 \rightarrow \frac{k_N^2}{2} = 3 \text{sech}^2(z)
\]

\[
\hat{\psi}(\pm \infty) = 0
\]

**Numerical Results Are Shown in the Following Page**

---

\[1\text{HINT}: \text{Show that the solution can be expressed as a linear combination of } P_2^{k_N}[\tanh(z)] \text{ and } Q_2^{k_N}[\tanh(z)], \text{ where } P_2^{k_N} \text{ and } Q_2^{k_N} \text{ are Legendre functions of degree 2 and order } k_N \text{ of the first and second kinds. To determine } k_N, \text{ note that } Q_2^{k_N}(1) = \infty \text{ and that}
\]

\[
P_2^{k_N}(0) = \frac{4}{\sqrt{\pi}} \cos \left[ \pi \left( 1 + \frac{k_N}{2} \right) \right] \Gamma \left( 3 + k_N/2 \right) \Gamma \left( 3 - k_N/2 \right) \quad \text{and} \quad (P_2^{k_N})'(0) = \frac{4}{\sqrt{\pi}} \sin \left[ \pi \left( 1 + \frac{k_N}{2} \right) \right] \Gamma \left( 3 + k_N/2 \right) \Gamma \left( 3 - k_N/2 \right).
\]

Finally, use the expressions \( P_2(s) = -3s \sqrt{1 - s^2} \) and \( P_2'(s) = 3(1 - s^2) \) to determine the eigenfunctions \( \hat{\psi}_N \).
Rayleigh’s equation for axisymmetric flows

For the stability analysis of parallel axisymmetric flows with basic velocity profile $\bar{U} = (U_x, U_r, U_\theta) = (U(r), 0, 0)$ we introduce modal perturbations of the form $(v'_x, v'_r, v'_\theta, p'/\rho) = [\hat{v}_x(r), \hat{v}_r(r), \hat{v}_\theta(r), \hat{p}(r)]e^{ik(x-ct)+in\theta}$. In the inviscid case, we use the linearized Euler equations to yield

\begin{align}
  i(kr\hat{v}_x + n\hat{v}_\theta) + \frac{d}{dr}(r\hat{v}_r) &= 0 \quad (1) \\
  (U - c)ik\hat{v}_x + \hat{v}_r \frac{dU}{dr} &= -ik\hat{p} \quad (2) \\
  (U - c)ik\hat{v}_r &= -\frac{d\hat{p}}{dr} \quad (3) \\
  (U - c)ik\hat{v}_\theta &= -\frac{in}{r}\hat{p}. \quad (4)
\end{align}

Eliminating $\hat{v}_\theta$ with use made of (1) and (4) provides

\begin{align}
  (U - c)ik\hat{v}_x - \frac{in^2}{kr^2}\hat{p} + \frac{U - c}{r} \frac{d}{dr}(r\hat{v}_r) &= 0,
\end{align}

which can be combined with (2) to yield

\begin{align}
  \frac{d}{dr}[(U - c)ik\hat{v}_r] + \left[\frac{1}{r} - \frac{2}{U - c} \frac{dU}{dr}\right](U - c)ik\hat{v}_r + \left(k^2 + \frac{n^2}{r^2}\right)\hat{p} &= 0,
\end{align}

finally giving the eigenvalue problem

\begin{align}
  \frac{d^2\hat{p}}{dr^2} + \left[\frac{1}{r} - \frac{2}{U - c} \frac{dU}{dr}\right] \frac{d\hat{p}}{dr} - \left(k^2 + \frac{n^2}{r^2}\right)\hat{p} &= 0; \quad \hat{p}(\infty) = 0 \text{ and } \hat{p}(0) \neq \infty
\end{align}

after substitution of (3). This equation has been used extensively to analyze the stability of round jets and axisymmetric wakes. The solution simplifies when the base velocity profile $U(r)$ is piecewise uniform, under which conditions the pressure in each uniform subdomain $i$ is given by

\begin{align}
  \hat{p}_i = A_i I_n(kr) + B_i K_n(kr),
\end{align}

where $I_n$ and $K_n$ are modified Bessel functions of order $n$. The interfaces separating the different subdomains are fluid surfaces with uniform pressure, thereby providing the jump conditions

\begin{align}
  [p] = 0 \quad \text{and} \quad \left[\frac{d\hat{p}}{dr}/(U - c)^2\right] = 0.
\end{align}

Batchelor & Gill (JFM 14, 1963) used this simplified approach to analyze the stability of a top-hat round jet of radius $a$ (i.e., with basic velocity profile $U = U_o$ for $r \leq a$ and $U = 0$ for $r > a$), obtaining

\begin{align}
  \frac{c_r}{U_o} &= \frac{1}{1 + L_n(ka)} \quad \text{and} \quad \frac{c_i}{U_o} = \frac{\sqrt{L_n(ka)}}{1 + L_n(ka)},
\end{align}

where $L_n(ka) = -[K_n(ka)I'_n(ka)]/[K'_n(ka)I_n(ka)] > 0$ (all modes are unstable).
Useful background information: Tollmien (1935) showed that the Rayleigh-Fjørtoft’s conditions are (plausibly) sufficient for instability of symmetric or monotonic profiles $U(z)$. In particular, he proved that

1. there is always a neutral disturbance given by $c_N = 0$, $k_N = 0$, $\hat{\psi}_N = U(z)$,
2. if $U'' = 0$ at $z = z_s$ then there exists a neutral disturbance with $k = k_N > 0$ and $c_N = U(z_s)$, and
3. for $k$ slightly less than $k_N$ there exist solutions with $c_i > 0$, and for $k$ slightly greater than $k_N$ there are no solutions with $c_i > 0$.

Let’s see how these ideas can be utilized to facilitate the investigation of the stability of a laminar wake described by the hyperbolic tangent profile $U(z) = \tanh(z)$.

- Show that the basic velocity profile satisfies Rayleigh-Fjørtoft’s conditions, necessary for instability.
- Determine the location of the inflection point $z = z_s$ and the value of $U_s = U(z_s)$.
- Show that, for the neutral disturbance, Rayleigh’s equation reduces to
  \[
  \frac{d^2\hat{\psi}}{dz^2} - (k_N^2 - 2 \cosh^{-2} z)\hat{\psi} = 0,
  \]
  which has the two independent solutions
  \[
  \hat{\psi} = k_N \cosh(k_N z) - \sinh(k_N z) \tanh z, \quad \hat{\psi} = k_N \sinh(k_N z) - \cosh(k_N z) \tanh z
  \]
- Determine the wave number $k_N$ and associated neutral eigenfunction $\hat{\psi}_N$.
- In the unstable range of wave numbers $0 < k < k_N$ one must use numerical integration to determine the solution. It is found that $c_i = 0$ for $0 < k < k_N$ while the growth rate of the disturbances $kc_i$ shows a maximum value $kc_i = 0.0948$ for $k = 0.4446$ ($c_i = 0.2132$).
- The numerical results can be used to compute the vorticity distribution of the perturbed flow (the sum of the base-flow vorticity and the vorticity perturbation)
  \[
  \omega_y = \frac{dU}{dz} + \epsilon e^{kc_i t} \text{Re} \left[ \left( \frac{d^2\hat{\psi}}{dz^2} - k^2 \hat{\psi} \right) e^{ki(x-ct)} \right]
  \]
  where $\epsilon$ is introduced as an arbitrary measure of the perturbation for illustrative purposes (results below are for $k = 0.4446$, $\epsilon = 0.2$, and $t = 0$).
Consider the wake formed downstream from a cylinder placed perpendicular to a uniform stream. To investigate its inviscid stability characteristics, we can use Rayleigh's equation together with the approximate piecewise linear representation

\[
U = \begin{cases} 
1 & \text{for } |z| \geq 1 \\
2(|z| - 1/2) & \text{for } 1/2 \leq |z| \leq 1 \\
0 & \text{for } |z| \leq 1/2 
\end{cases}
\]

for the velocity profile. Determine the eigenvalue relation \(c(\alpha)\) in explicit form, considering separately the sinuous and varicose modes. Plot the variation of \(c_r\) and \(c_i\) with \(k \geq 0\). Discuss your results. Do they (qualitatively) agree with the experimental observations regarding the Von Karman vortex street?

\[
\beta_1 e^{-kz} = A_2 e^{kz} + \beta_2 e^{-kz} \quad (1)
\]

\[
\begin{align*}
- (1-c) \beta_1 e^{-kz} &= (1-c) \left( A_2 e^{-kz} + \beta_2 e^{kz} \right) - Z \left( A_2 e^{kz} + \beta_2 e^{-kz} \right) \\
- c \left( A_2 e^{kz/2} - \beta_2 e^{-kz/2} \right) - Z \left( A_2 e^{kz/2} + \beta_2 e^{-kz/2} \right) &= - c A_2 \left( e^{kz/2} - e^{-kz/2} \right) \\
A_2 e^{kz/2} + \beta_2 e^{-kz/2} &= A_3 \left( e^{kz/2} - e^{-kz/2} \right) \\
A_2 e^{kz/2} + \beta_2 e^{-kz/2} &= A_3 \left( e^{-kz/2} - e^{kz/2} \right)
\end{align*}
\]

From (1) & (2):

\[
\beta_2 = \left[ (1-c) k - 1 \right] A_2 e^{2k} 
\]

From (3) & (4):

\[
A_2 \left[ ck + e^{kz} \right] = \beta_2 \left[ ck - (1 + e^{-kz}) \right] 
\]

From (4) & (5):

\[
A_2 \left[ ck - e^{-kz} \right] = - \beta_2 \left[ ck - (1 + e^{-kz}) \right] 
\]

Combining (5) & (6) and solving for

\[
C = C_2 = \frac{k + e^{2k} - e^{-2k}}{2k} + \frac{i}{2k} \left[ \sqrt{4(k+1+ke^{2k}e^{-2k})-(k+e^{2k}e^{-2k})^2} \right]
\]

\[
C = C_2 = \frac{k - e^{2k}e^{2k} - e^{-2k}}{2k} + \frac{i}{2k} \left[ \sqrt{4(1-k)(1+e^{2k}e^{-2k})-(k+e^{2k}e^{-2k})^2} \right]
\]

The growth rate of the sinuous

Mode is larger for all \(k\), denoting

that this is the prevailing instability mode,

in agreement with experimental observations (the alternative vortex spreading of the Von-Karman street produces a sinuous pattern).
Combustion in nonpremixed systems requires the mixing of the reactants in the mixing layers that separate the fuel and air feed streams. Mixing is greatly enhanced when the mixing layers are unstable and display transition to turbulent flow. In most applications, the prevailing Mach number is small and the effect of gravity is negligible, so that to investigate the stability response of isothermal fuel-air mixing layers (and planar fuel jets) to inviscid two-dimensional perturbations we can begin by writing the Euler equations in the buoyancy-free form

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \\
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{\nabla p}{\rho} \\
\frac{\partial Y}{\partial t} + \vec{v} \cdot \nabla Y = 0
\]

which must be integrated together with the low-Mach-number form of the equation of state

\[
\frac{\rho_{\text{AIR}}}{\rho} = 1 + \left( \frac{\rho_{\text{AIR}}}{\rho_{\text{FUEL}}} - 1 \right) Y,
\]

where \( Y \) is the mass fraction of the fuel.

- Show that one can replace the conservation equations for mass and fuel by the alternative equations

\[ \nabla \cdot \vec{v} = 0 \quad \text{and} \quad \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = 0, \]

and that the basic solution \( \vec{v} = U(z) \hat{e}_x, \rho = \tilde{\rho}(z), \) and \( p = P = \text{constant} \) satisfies the conservation equations.

- Introduce normal modes and derive the eigenvalue problem

\[ (U - c) \left[ \frac{d}{dz} \left( \frac{d \hat{v}_z}{dz} \right) - k^2 \tilde{\rho}_0 \hat{v}_z \right] - \frac{d}{dz} \left( \frac{dU}{dz} \right) \hat{v}_z = 0; \quad \hat{v}_z = 0 \text{ at } z = z_1, z_2 \]

- Derive the modified Rayleigh-Fjørtoft’s conditions for these variable-density systems.

The problem admits simplified solutions when the velocity and density profiles are piecewise linear and piecewise uniform, respectively, so that \( \hat{v}_z = A_t e^{kz} + B_t e^{-kz} \) in each subdomain \( t \).

- Use the modified Rayleigh’s equation given above as a starting point to determine the jump conditions that need to be satisfied at the boundaries between subdomains. Interpret your results in physical terms.

- The mathematical framework developed can be used to analyze the temporal inviscid stability of a planar fuel jet discharging into stagnant air, by approximating the velocity and density profiles by the linear functions \( U = 0 \) and \( \tilde{\rho} = 1 \) for \( |z| > 1 \) and \( U = 1 - |z| \) and \( \tilde{\rho} = S \) for \( |z| < 1 \). Consider separately the sinuous and varicose modes.

- Discuss the results corresponding to hydrogen (light) jets with \( S < 1 \) and dodecane (heavy) jets with \( S > 1 \).
THE EQUATION OF STATE INDICATES THAT
\[ \frac{\partial p}{\partial \alpha} \frac{\partial s}{\partial \alpha} = \left( \frac{\partial s}{\partial \alpha} - \frac{\partial p}{\partial \alpha} \right) \frac{\partial \alpha}{\partial s} = 0 \]

Which can be used to reduce the continuity equation
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla \rho = 0 \Rightarrow \nabla \cdot \mathbf{v} = 0 \]

\[ \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{v}) = - \frac{\partial p}{\partial s} \Rightarrow \frac{\partial \mathbf{v}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u} - \frac{\partial p}{\partial s} = 0 \]

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = 0 \]

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = 0 \]

\[ \frac{\partial \mathbf{v}}{\partial t} = - \frac{1}{s} \frac{\partial \mathbf{v}}{\partial s} \]

Combining the first two equations
\[ -(U-C) \frac{\partial \mathbf{v}}{\partial z} + \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial z} - \frac{\partial \mathbf{v}}{\partial z} \cdot \frac{\partial \mathbf{u}}{\partial z} = 0 \]

which can be used to eliminate the pressure in the 2-component of the momentum equation
\[ -\frac{s}{k^2}(U-C) \frac{\partial \mathbf{v}}{\partial z} = \frac{\partial s}{\partial z} \left[ -(U-C) \frac{\partial \mathbf{v}}{\partial z} + \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial z} \right] + \frac{s}{k^2} \left[ -(U-C) \frac{\partial \mathbf{v}}{\partial z} + \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial z} \right] \]

\[ (U-C) \left[ \frac{\partial}{\partial z} \left( \frac{s}{k^2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial z} \right) \right] - \frac{\partial s}{\partial z} \left( \frac{s}{k^2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial z} \right) \mathbf{v} = 0 \]

This last equation can be integrated across the fluid domain after multiplication by \( U-\alpha \) to give
\[ \frac{\partial s}{\partial z} \left( \frac{s}{k^2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial z} \right) \mathbf{v} = 0 \]

The imaginary part reads
\[ C \int_{z_1}^{z_2} \frac{\partial s}{\partial z} \left( \frac{s}{k^2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial z} \right) \mathbf{v} \cdot d\mathbf{z} = 0 \]

If \( C \) to then \( \frac{\partial s}{\partial z} \left( \frac{s}{k^2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial z} \right) \) must change sign, there has to be (at least) one point \( z_3 \) for \( z_1 \leq z_3 \leq z_2 \)

At which \( \frac{\partial s}{\partial z} \left( \frac{s}{k^2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial z} \right) = 0 \)

The real part
\[ \left( C_0 - U_0 \right) \int_{z_1}^{z_2} \frac{\partial s}{\partial z} \left( \frac{s}{k^2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial z} \right) \mathbf{v} \cdot d\mathbf{z} = 0 \]

yields
\[ \int_{z_1}^{z_2} \frac{\partial s}{\partial z} \left( \frac{s}{k^2} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial z} \right) \mathbf{v} \cdot d\mathbf{z} = 0 \]

across the flow.
\[
\frac{d}{dt}\left\{ \frac{S}{\bar{S}} \left[ (U-c) \frac{d\hat{v}_2}{dt} - \frac{dU}{dt} \hat{v}_2 \right] \right\} - k^2 (U-c) \bar{S} \hat{v}_2 = 0
\]

**INTEGRATING ONCE**

\[
\bar{S}\left[ (U-c) \frac{d\hat{v}_2}{dt} - \frac{dU}{dt} \hat{v}_2 \right] = k^2 \left[ (U-c) \bar{S} \hat{v}_2 \right] d\tau + \text{const} = 0 
\Rightarrow
\]

\[
\left\{ \left[ \bar{S} \left[ (U-c) \frac{d\hat{v}_2}{dt} - \frac{dU}{dt} \hat{v}_2 \right] \right] \right\} = 0
\]

**CONTINUITY OF PRESSURE**

**REARRANGING AND INTEGRATING AGAIN**

\[
\frac{\hat{v}_2}{U-c} = k^2 \int \frac{1}{\bar{S} (U-c)^2} \int S (U-c) \bar{S} \hat{v}_2 d\tau' d\tau + \int \text{const} \frac{\bar{S}}{\bar{S} (U-c)^2} d\tau = 0
\Rightarrow
\]

\[
\left\{ \left[ \frac{\hat{v}_2}{U-c} \right] \right\} = 0 
\Rightarrow
\]

**THE INTERFACE IS A FLUID SURFACE**

**VARICOSE MODE**

\[
B_1 e^{-kz} + A_2 (e^{kz} - e^{-kz}) + A_2 (e^{kz} - e^{-kz}) = B_1 e^{-kz} = A_2 (e^{kz} - e^{-kz}) + A_2 (e^{kz} - e^{-kz})
\]

\[
+ckB_1 e^{-k} = \left[ -cA_2 k (e^{kz} + e^{-kz}) + A_2 (e^{kz} - e^{-kz}) \right] S
\]

\[
ck = \frac{S' (1 - e^{-2k})}{1 + S + e^{-2k} (S-1)}
\]

**ALWAYS STABLE**

**SINUOUS MODE**

\[
B_1 e^{-kz} + \frac{A_2 e^{kz} + B_2 e^{-kz}}{B_2 e^{kz} + A_2 e^{-kz}} = B_1 e^{-kz} + B_2 e^{-kz}
\]

\[
+ckB_1 e^{-k} = \frac{B_1 e^{-k} + A_2 e^{kz} + B_2 e^{-kz}}{B_2 e^{kz} + A_2 e^{-kz}} = A_2 e^{kz} + B_2 e^{-kz}
\]

\[
+ckB_1 e^{-k} = \frac{S'}{S} \left[ -ck (A_2 e^{kz} - A_2 e^{-kz}) + A_2 e^{kz} + B_2 e^{-kz} \right]
\]

\[
[ck(S-1) + S][ck-k-1] = e^{+2k} [ck(S+1) - S][ck-k-1]
\]

\[
(ck)^2 \left\{ S \sinh(k) + \cosh(k) \right\} - ck \left\{ (S-1) \sinh k + k \cosh k \right\} + S \left( k \cosh k - \sinh k \right) = 0
\]

\[
ck = \frac{(S-1) \sinh k + k \cosh k}{2 [S \sinh(k) + \cosh(k)]} = \frac{\sqrt{4S(k \cosh k - \sinh k)(S \sinh(k) + \cosh(k)) - [S \sinh(k) + \cosh(k)]}}{2 [S \sinh(k) + \cosh(k)]}
\]
Problem 1 (30 points): The piecewise linear profile

\[ U(z) = \begin{cases} 
2bz & 0 \leq z < 1/2 \\
1 + 2(1 - b)(z - 1) & 1/2 \leq z < 1 \\
1 & 1 \leq z 
\end{cases} \]

where \( b \leq 1 \) is a positive constant, is proposed as an approximate representation of the velocity across a boundary layer subject to an adverse pressure gradient. In order to investigate the temporal inviscid stability of this flow, integrate Rayleigh’s equation to obtain an explicit expression for the dispersion relation \( c = c(k; b) \), where \( k \in \mathbb{R}^+ \) is the wavenumber. Show that the boundary layer is neutrally stable for \( b > 1/2 \), that is, \( c_i = 0 \) for all positive values of \( k \). Plot the curves \( \omega_i(k) = kc_i(k) \) for \( b = 1/3 \) and \( b = 1/5 \). Discuss your results.
Problem 1

This models a NL profile with an induction point. A Rayleigh's criterion for instability is satisfied:

\[ b > 1/2 \quad \hat{\nu}_2 = B_1 e^{-k_2} \]

\[ 1 > b > 1/2 \quad \hat{\nu}_2 = A_2 e^{k_2} + B_2 e^{-k_2} \]

\[ b < 1/2 \quad \hat{\nu}_2 = A_3 (e^{k_2} - e^{-k_2}), \text{ st. } \hat{\nu}_c(0) = 0. \]

Apply jump conditions, \( \Gamma (0-c) \frac{\partial \phi}{\partial z} + A_2 e^{k_2} = A_2 e^{-k_2} \), \( 0 = \hat{\nu}_2(0-c) \), \( 0 = \hat{\nu}_c(0) = 0 \) at \( z = 1/2, 1/10 \):

\[
\begin{align*}
1) \quad B_1 e^{-k} &= A_2 e^k + B_2 e^{-k} \\
2) \quad -(1-c)k B_1 e^{-k} &= (1-c)k (A_2 e^k - B_2 e^{-k}) - 3(1-b)(A_2 e^k + B_2 e^{-k}) \\
3) \quad (b-c)k(A_2 e^{k/2} - B_2 e^{-k/2}) - 2(1-b)(A_2 e^{k/2} + B_2 e^{-k/2}) &= (b-c)k A_3 (e^{k/2} - e^{-k/2}) - 2b A_2 (e^{k/2} - e^{-k/2}) \\
4) \quad A_2 e^{k/2} + B_2 e^{-k/2} &= A_2 (e^{k/2} - e^{-k/2})
\end{align*}
\]

Combining (3) \& (2): 
\[ A_2 e^{k} (2ck + 2(1-k-b)) = -2(1-b)R_2 e^{-k} \]

Combining (3) \& (4): 
\[ -A_2 e^{k} [2(b-c)ke^{-k} + 2(b-1)(1-e^{-k})] = R_2 + \]

we obtain, finally:

\[ (2ck + 2(1-k-b))(2ck - 2bk + 2(b-1)(1-e^{-k}))e^{k} = 2e^{-k}(1-b)[2ck e^{-k} - 2bke^{-k} + 2(b-1)(1-e^{-k})] \]

Now:

\[ \text{if } b = 1/2 \Rightarrow c = A e^{-2k} \quad, \quad C_i = 0 \]

\[ b > 1/2 \Rightarrow C_i = 0 \]

For \( b < 1/2 \), see remarks.
Problem 2 (70 points): As discussed in class, the inviscid stability of unconfined axisymmetric constant-density flows with basic velocity profile \( U = U(r)e_x \) can be examined by solving the eigenvalue problem

\[
\frac{d^2 \hat{p}}{dr^2} + \left( \frac{1}{r} - \frac{2}{U - c} \frac{dU}{dr} \right) \frac{d\hat{p}}{dr} - \left( k^2 + \frac{n^2}{r^2} \right) \hat{p} = 0, \quad |\hat{p}(0)| < \infty, \quad \hat{p}(\infty) = 0,
\]

If the basic profile contains a finite number of discontinuities, then the jump conditions

\[
[\hat{p}] = 0, \quad \left[ \frac{1}{(U - c)^2} \frac{d\hat{p}}{dr} \right] = 0,
\]

must be imposed at each discontinuity.

- Investigate the inviscid stability of an axisymmetric wake approximated by the piecewise uniform profile \( U(r) = \beta U_o \) for \( r \leq a \) and \( U(r) = U_o \) for \( r > a \), where \( U_o \) is a positive constant velocity, \( a \) is a characteristic length, and \( \beta < 1 \) is a constant. Could you have anticipated this result from the results of Batchelor and Gill (1963) for the dispersion relation of a top-hat jet?

Consider now the stability of axisymmetric parallel flows with variable density (e.g. a light gas jet discharging into a heavier gas atmosphere). For isothermal flow we can begin by writing the Euler equations in the buoyancy-free form

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0, \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p, \quad \frac{\partial Y}{\partial t} + \mathbf{v} \cdot \nabla Y = 0,
\]

which must be integrated together with the low-Mach-number form of the equation of state

\[
\rho_\infty / \rho = 1 + (S - 1)Y.
\]

Here \( Y \) is the mass fraction of the jet gas and \( S = \rho_\infty / \rho_j \) is the ambient-to-jet density ratio.

- Show that one can replace the conservation equations for mass and fuel mass fraction by the alternative equations

\[
\nabla \cdot \mathbf{v} = 0, \quad \text{and} \quad \frac{D\rho}{Dt} = 0,
\]

where \( D/Dt \) represents the material derivative. Verify that the base-flow solution \( \mathbf{v} = U(r)e_x, \rho = \bar{\rho}(r), \) and \( p = P = \text{constant} \) satisfies the conservation equations.

- Introduce normal modes proportional to \( \exp \left[ ik(x - ct) + in\theta \right] \) and deduce a single differential equation for the pressure perturbation. Show that this equation may be expressed as

\[
\frac{d^2 \hat{p}}{dr^2} + \left( \frac{1}{r} - \frac{2}{U - c} \frac{dU}{dr} \right) \frac{d\hat{p}}{dr} - \left( k^2 + \frac{n^2}{r^2} \right) \hat{p} = 0,
\]

Derive the jump conditions satisfied by \( \hat{p} \) across a discontinuity.
The above formulation can be used to analyze the temporal inviscid stability of a round fuel jet discharging into stagnant air, by approximating the (nondimensional) velocity and density profiles by the piecewise uniform functions $U = 0$ and $\rho = 1$ for $r > 1$ and $U = 1$ and $\rho = S$ for $r < 1$. Find an expression for the eigenvalue relation $D(c, k, n; S) = 0$ and discuss your results.

Indicate how the differential equation satisfied by $\hat{p}$ and the corresponding jump conditions would be modified if a gravitational body force in the streamwise direction is included (i.e. as for a light jet pointing upwards). In your analysis replace the original momentum equation by

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + (\rho_\infty - \rho) g \mathbf{e}_x.$$

Show all the steps involved in the derivation.
\[ \frac{d}{dr}(r^2 \phi') - (k^2 + \kappa^2/r^2) \phi^2 = 0 \quad \Rightarrow \quad \phi^2 = \frac{L}{c^2} \quad \text{(1)} \]

\[ \phi^2(1) = A_1 \text{In}(ka) + B_1 \text{Kn}(ka) \]

\[ \phi^2(2) = A_2 \text{In}(ka) + B_2 \text{Kn}(ka) \]

\[ \Delta \phi^2 = \frac{L}{c^2} = 0 \quad \text{at} \quad r = a \]

Thus, on dividing (1)/(2) one finds,

\[ c = \frac{U_0}{V_0} \left( \sqrt{\frac{L}{c^2}} \pm \frac{i \beta (1-\beta)}{\sqrt{L}} \right) \quad \Rightarrow \quad \sqrt{\frac{L}{c^2}} = \frac{\ln \alpha}{1 + \ln \alpha} \]

\[ \frac{\xi}{U_0} = \frac{L \alpha + \beta}{1 + \ln \alpha} \quad \Rightarrow \quad \frac{\xi}{U_0} = \pm \frac{\ln \alpha}{1 + \ln \alpha} > 0, \; \forall \alpha, \; \delta \]

the flow is unsteady, unstable and inviscid perturbation.

Note that the prop of Galilean transformations:

\[ \begin{align*}
U_0 & \quad \rightarrow \quad -U_0 \\
V_0 & \quad \rightarrow \quad -V_0 \\
\Delta U_0 & \quad \rightarrow \quad -\Delta U_0 \\
\Delta V_0 & \quad \rightarrow \quad -\Delta V_0 \\
\alpha & \quad \rightarrow \quad -\alpha
\end{align*} \]

For \( \beta = 0 \), Ratibek Ct. Gili's result:

\[ \begin{align*}
\frac{U_0 - \zeta}{U_0} & = \frac{1}{1 + \ln \alpha} \quad \Rightarrow \quad \frac{\xi}{U_0} = \frac{L \ln \alpha}{1 + \ln \alpha} \\
\frac{\pm \xi}{U_0} & = \pm \frac{\ln \alpha}{1 + \ln \alpha}
\end{align*} \]

\[ \text{which would be anticipated from (2) \ for } \beta = 0. \quad \text{That is, a symmetric wake and yet have the same dispersion relation!} \]
In general, to the transformation \( U \rightarrow \alpha U + \beta = : W \), it is convenient to consider the eigenvalue transformation \( \alpha \rightarrow \alpha c + \beta = : \alpha' \). Substituting this into the original PDE yields:

\[
\alpha_0 \frac{d}{dr} \left( r \frac{d^2 \alpha'}{dr^2} \right) - \left( k^2 + \eta^2 \frac{1}{r^2} \right) \alpha' - \frac{2}{r-c} \frac{dU}{dr} \frac{\partial p}{\partial r} = 0
\]

\[
\frac{2}{W-\lambda - \alpha c} \frac{dW}{dr} \frac{p}{c'} = \frac{2}{W-\lambda - \alpha c} \frac{dw}{dr},
\]

resulting in the modified eigenvalue problem:

\[
\alpha_0 \frac{d}{dr} \left( r \frac{d^2 \alpha'}{dr^2} \right) - \left( k^2 + \eta^2 \frac{1}{r^2} \right) \alpha' - \frac{2}{r-c} \frac{dW}{dr} \frac{\partial p}{\partial r} = 0 \quad (\text{p}(r) \equiv \infty \text{ on } r \in (0, \infty))
\]

with jump conditions \( D\hat{p} = \Delta \left[ (W-c')^{-2} p' \right] = 0 \) across discontinuities. Notice that \( \text{Im}(c') \) may change sign due to \( \alpha \), but this is no problem since for every \( c', c' \) is also an eigenvalue of \( \alpha \).

\[\text{i)} \quad \text{Since } Y = \hat{Y}(\rho), \quad D\hat{V}/Dt = 0 \Rightarrow \frac{\partial \hat{V}}{\partial \rho} \frac{D\rho}{Dt} = 0. \quad \text{Provided}
\]

\[\frac{\partial \hat{V}}{\partial \rho} \neq 0 \text{ and zero, we conclude that } \frac{D\rho}{Dt} = 0. \quad \text{Therefore, continuity reads:}
\]

\[
\frac{D\rho}{Dt} = -\nabla \cdot \hat{u} = 0
\]

\[
\hat{u} = 0(r) \hat{e}_x, \quad \rho = \hat{\rho}(r), \quad \hat{p} = \hat{p} \quad \text{satisfy the Euler equations:}
\]

\[
\nabla \cdot \hat{u} = 0 \Rightarrow \hat{\rho} \frac{\partial \hat{u}}{\partial x} + \frac{1}{2} \hat{\eta} \frac{\partial A}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial \hat{v}}{\partial r} \right] = 0
\]

\[
\frac{\rho}{\hat{\rho}} \frac{\partial \hat{v}}{\partial t} + \frac{\partial \hat{u}}{\partial x} \frac{\partial \hat{v}}{\partial x} = -\frac{\nabla \hat{p}}{\hat{\rho}}
\]

\[
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \hat{u} \cdot \nabla \rho = 0(r) \frac{\partial \rho}{\partial x} = 0
\]
iii) The linearized equations are:

(a) \( \nabla \cdot \nabla' = 0 \),

(b) \( \rho \frac{d\hat{\rho}}{dt} + p' \frac{\hat{\rho} \cdot \nabla \hat{\rho}}{\hat{\rho}} + p' \cdot \nabla \hat{\rho} + \hat{\rho} \cdot \nabla \hat{\rho}' = - \nabla \rho' \)

(c) \( \frac{d\hat{V}}{dt} + \hat{V} \cdot \nabla \hat{V} + \frac{\hat{U} \cdot \nabla \hat{V} \hat{V}'}{\hat{U} \cdot \nabla \hat{V}} \)

(d) \( i k (x - ct) + in \rho = 0 \)

From (17), solve for \( \hat{\rho} = \hat{\rho} \) and

From (16), solve for \( \hat{\hat{v}} \) and

Introduce \( \hat{\hat{v}}, \hat{\rho} \) in (14) as

Introduce of \( \hat{\hat{v}} \). Combine with

(14) to finally find:

(19) \[ \frac{d^2 \hat{\rho}}{dr^2} + \left( \frac{1}{r} - \frac{2}{r} \right) \frac{d\hat{\rho}}{dr} - \left( k^2 + \frac{n^2}{r^2} \right) \hat{\rho} = 0 \]

where both \( \hat{\rho} \) and \( \hat{\rho} \) may contain a finite number of discontinuities throughout the fluid domain. To derive jump conditions across these discontinuities, note that (19) may be written as:

(20) \[ \frac{d}{dr} \left( \frac{1}{\rho(u-c)^2} \frac{d\hat{\rho}}{dr} \right) - \frac{r}{\rho(u-c)^2} \left( k^2 + \frac{n^2}{r^2} \right) \hat{\rho} = 0 \]

Integrate (20) through \( -\infty < r < \infty \) to find:

(21) \[ \Delta \frac{1}{\rho(u-c)^2} r \frac{d\hat{\rho}}{dr} = \int_{-\infty}^{\infty} \frac{r dr}{\rho(u-c)^2} \left( k^2 + \frac{n^2}{r^2} \right) \to 0 \quad \text{as} \quad r \to 0 \]

Thus:

\[ \left[ \frac{1}{\rho(u-c)^2} r \frac{d\hat{\rho}}{dr} \right] = 0 \]; where \( \Delta(\cdot) = [\Delta] \) indicates jump.

(22) Integrate (20) twice to find:

(23) \[ \hat{\rho}'(r) = A_1 + \int_{-\infty}^{r} A_2 \rho(u-c)^2 + \rho(u-c)^2 \int_{-\infty}^{x} \frac{r dr}{\rho(u-c)^2} \left( k^2 + \frac{n^2}{r^2} \right) dr \]

Therefore \( \Delta \hat{\rho} = 0 \) emerges as the second jump condition for \( \hat{\rho} \). In (23),

\( A_1, A_2 \) are constants of integration.
iv) We apply these ideas to: 

Since \( \partial^2 U/\partial r^2 = 0 \) (except at \( r = a \)), we must have:

\[
\frac{d}{dr} \left( r \frac{d\rho}{dr} \right) + \frac{1}{r} \frac{d\rho}{dr} + \left( k^2 + \frac{n^2}{r^2} \right) \rho = 0, \quad r > 0
\]

along with jump conditions:

\[
\begin{align*}
\rho &= A K_n(kr) \\
\rho &= B I_n(kr)
\end{align*}
\]

v) We recover Ratchelor's result for \( S = 1 \). In (26), \( \alpha = ka \) and \( ln \) is defined as in eqs. (2) - (3).

\[
S(1-c)^2 + \ln(\alpha) c^2 = 0 \Rightarrow \left\{ \begin{align*}
\alpha &= \frac{S}{(\ln(\alpha) + S)} \\
\alpha &= \pm \frac{\sqrt{SLn(\alpha)}}{\ln(\alpha) + S}
\end{align*} \right. \text{ always unstable! (What happens when \( S \to \infty, \) \( S \to 0 \)??)}
\]

When adding buoyancy effects, x-momentum is modified as follows:

\[
\rho(U-c) \frac{d\rho}{dr} + \rho \frac{d^2 U}{dr^2} = -i\kappa \rho + \rho g
\]

and, from (18): 

\[ \rho = i\kappa \frac{dU}{dr} \frac{1}{k(U-c)} \]

When combined with (16) gives the modified Rayleigh's equation:

\[
\left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d\rho}{dr} \right) \right] - \left\{ \frac{2}{U-c} \frac{dU}{dr} + \frac{1}{\rho} \frac{d\rho}{dr} - \frac{g}{i\kappa(U-c)^2} \frac{1}{\rho} \frac{d\rho}{dr} \right\} \frac{d^2 \rho}{dr^2} - \left( k^2 + \frac{n^2}{r^2} \right) \rho = 0
\]
Problem 3 (Extra credit 20 points): Consider the viscous stability to planar perturbations of planar Poiseuille flow between two infinite parallel plates separated by a distance $2h$. Scaling the problem with the channel semi-width $h$ and the peak velocity $U_m$ reduces the Orr-Sommerfeld equation to

$$
\left[(U - c) \left(D^2 - k^2\right) - D^2U - (ik R)^{-1} \left(D^2 - k^2\right)^2\right] \hat{v}_z = 0, \quad -1 < z < 1,
$$

where $D = d/dz$, $R = U_m h/\nu$ is the Reynolds number, $U(z) = 1 - z^2$ is the nondimensional base velocity profile, $k$ is the streamwise wavenumber, $i^2 = -1$, and $c = c_r + ic_i$ is the eigenvalue. The homogeneous boundary conditions $\hat{v}_z = D\hat{v}_z = 0$ must be imposed at the walls, $z = \pm 1$.

- Show by numerical integration of the above eigenvalue problem that the margin of stability corresponds to $(R_c, k_c) = (5772, 1.02)$ [see link for reference]. To that end, obtain the spectrum $(c_r, c_i)$ for $(R, k) = (5772, 1.02)$ (include at least 30 – 50 eigenvalues) and compare the result with the spectra corresponding to $k = 1.1$ and $k = 0.95$ for $R = 5772$ and with the spectra corresponding to $R = 5800$ and $R = 5750$ for $k = 1.02$.

- Determine the growth rate $kc_i$ as a function of $k$ for different representative values of $R > R_c$ up to $R \to 10^6$ and use your results to plot the marginal curve $C(R, k) = 0$ bounding the region of unstable flow on the plane $R - k$ [you should obtain Drazin’s figure 8.8 (a)].

Comment on your choice of numerical method and on the difficulties you found for large values of $R$. Provide a hardcopy of your codes as an appendix.
PROBLEM 3

a) Computed spectrum for marginal conditions \((Re, k) = (5772, 1.02)\). The corresponding marginal (Airy) eigenvalue is highlighted in red.

\[ Re = 5772, \quad k = 1.02 \]

b) Marginal curve in the \(Re - k\) plane