Problem 1 (1.2 in Acheson): An ideal fluid is rotating under gravity \( g \) with constant angular velocity \( \Omega \), so that relative to fixed cartesian axes \( \mathbf{v} = (-\Omega y, \Omega x, 0) \). We wish to find the surfaces of constant pressure, and hence the surface of a uniformly rotating bucket of water (which will be at atmospheric pressure).

‘By Bernoulli,’ \( \frac{p}{\rho} + \frac{1}{2} \mathbf{v}^2 + gz \) is constant, so that the constant pressure surfaces are

\[
z = \text{constant} - \frac{\Omega^2}{2g} (x^2 + y^2).
\]

But this means that the surface of a rotating bucket of water is at its highest in the middle. What is wrong?

Write down the Euler equations in component form, integrate them directly to find the pressure \( p \), and hence obtain the correct shape for the free surface.

Solution: It is true that for steady inviscid flow of a constant-density fluid \( \frac{p}{\rho} + \frac{1}{2} \mathbf{v}^2 + gz = C \) along every streamline, but the constant \( C \) may be different along different streamlines, which is precisely what happens here. Since the velocity is known, we may determine the pressure directly by integrating the system of equations

\[
0 = -\frac{\partial p}{\partial z} - \rho g, \quad -\rho \Omega^2 x = -\frac{\partial p}{\partial x}, \quad -\rho \Omega^2 y = -\frac{\partial p}{\partial y}
\]

to give \( p = \frac{1}{2} \rho \Omega^2 (x^2 + y^2) - \rho gz + p_0 \), where the \( p_0 \) is the value of the pressure at the origin of the cartesian coordinate system. Correspondingly, the surface of the rotating bucket, where \( p = p_a \), is the paraboloid

\[
z = \frac{p_0 - p_a}{\rho g} + \frac{\Omega^2}{2g} (x^2 + y^2).
\]

Problem 2 (from 1.4 in Acheson): Take the Euler momentum equation for an incompressible fluid of constant density

\[
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p - \rho g \mathbf{e}_z,
\]

cast it into an appropriate form, and perform suitable operations on it to obtain the energy equation:

\[
\frac{d}{dt} \int_V \frac{1}{2} \rho \mathbf{v}^2 \, dV = -\int_S (p + \rho gz + \frac{1}{2} \rho \mathbf{v}^2) \mathbf{v} \cdot \mathbf{n} \, dS,
\]

where \( V \) is the volume enclosed by a fixed closed surface \( S \) drawn in the fluid, with \( \mathbf{n} \) denoting the corresponding normal unit vector pointing outwards.

Solution: We begin by writing the momentum equation in the form

\[
\rho \frac{\partial \mathbf{v}}{\partial t} + \nabla (p + \rho gz + \frac{1}{2} \rho \mathbf{v}^2) - \rho \mathbf{v} \wedge (\nabla \wedge \mathbf{v}) = 0.
\]

By taking the dot product with \( \mathbf{v} \) and using the condition \( \nabla \cdot \mathbf{v} = 0 \) we may derive a conservation equation for the kinetic energy:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho \mathbf{v}^2 \right) + \nabla \cdot \left[ (p + \rho gz + \frac{1}{2} \rho \mathbf{v}^2) \mathbf{v} \right] = 0.
\]

Integrating now the above equation in a volume \( V \) enclosed by a fixed closed surface \( S \) and using \( \int_V \frac{\partial}{\partial t} \mathbf{v} \, dV = \frac{d}{dt} \int_V \mathbf{v} \cdot dV \) for the first term along with Gauss theorem \( \int_V \nabla \cdot \mathbf{v} \, dV = \int_S \mathbf{v} \cdot \mathbf{n} \, dS \) for the second term provides the desired result.