Problem 1: Consider the equilibrium temperature distribution $u_e(x)$ along a one-dimensional rod of length $L$ and cross-sectional area $A$. The material of the rod has a constant thermal conductivity $K_o$. A volumetric heating rate $Q = Q_o \exp(x/L)$ is applied inside the rod, where $x$ is the distance from the left end and $Q_o$ is a constant (with dimensions of energy per unit volume per unit time). The two ends of the rod are kept at constant temperature, with $u(0, t) = u_0$ and $u(L, t) = u_L$.

- Determine the equilibrium temperature distribution $u_e$.
- Obtain the associated heat fluxes at both ends $\phi_0 = -K_o \frac{du_e}{dx}(0)$ and $\phi_L = -K_o \frac{du_e}{dx}(L)$.
- Calculate the total thermal energy generated inside the rod $\int_0^L Q(x) A \, dx$.
- What is the equation relating $\int_0^L Q(x) A \, dx$, $\phi_0$, and $\phi_L$?

Problem 2: In the absence of internal heat sources, heat conduction in a rod of length $L$ with uniform density $\rho$, specific heat $c$, and thermal conductivity $K_o$ is determined by integration of

$$\rho c \frac{\partial u}{\partial t} = K_o \frac{\partial^2 u}{\partial x^2}$$

with initial condition $u(x, 0) = u_i(x)$ and appropriate boundary conditions at both ends. Investigate the existence of equilibrium temperature distributions $u_e(x)$ for $t \to \infty$ in the following cases:

- $u(0, t) = u_0, -K_o \frac{du}{dx}(L, t) = \phi_L$
- $u(0, t) = u_0, -K_o \frac{du}{dx}(L, t) = H[u(L, t) - u_B]$
- $-K_o \frac{du}{dx}(0, t) = \phi_1, -K_o \frac{du}{dx}(L, t) = \phi_1$

where $u_0$, $\phi_L$, $H$, $u_B$, and $\phi_1$ are given positive constants.

Problem 3: Obtain the equilibrium temperature distribution $u_e(x)$ corresponding to a bar of length $L$ with constant thermal conductivity $K_o$ subject to a volumetric cooling rate $Q = -C^2(K_o/L^2)u$ proportional to the temperature $u$, with $C$ being a positive dimensionless constant. Assume that both ends are kept at the same constant temperature $u(0, t) = u(L, t) = u_o$. Evaluate the heat fluxes at both ends and verify that $\int_0^L Q(x) \, dx = \phi_L - \phi_0$.