Problem 1

Consider the equilibrium temperature distribution \( u_e(x) \) along a one-dimensional rod of length \( L \) and cross-sectional area \( A \). The material of the rod has a constant thermal conductivity \( K_0 \). A volumetric heating rate \( Q = Q_o \exp(x/L) \) is applied inside the rod, where \( x \) is the distance from the left end and \( Q_o \) is a constant (with dimensions of energy per unit volume per unit time). The two ends of the rod are kept at constant temperature, with \( u(0, t) = u_0 \) and \( u(L, t) = u_L \).

- Determine the equilibrium temperature distribution \( u_e \).
- Obtain the associated heat fluxes at both ends \( \phi_0 = -K_0 \frac{du_e}{dx}(0) \) and \( \phi_L = -K_0 \frac{du_e}{dx}(L) \).
- Calculate the total thermal energy generated inside the rod \( \int_0^L Q(x)A \, dx \).
- What is the equation relating \( \int_0^L Q(x)A \, dx \), \( \phi_0 \), and \( \phi_L \)?

Solution 1:

We begin with the 1D Heat Equation

\[
\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q
\]

- Inserting the equilibrium distribution \( u_e = u_e(x) \) into the Heat Equation yields

\[
0 = K_0 u_e'' + Q_0 \exp(x/L)
\]

where the primes denote differentiation with respect to \( x \). Integrating twice we find

\[
u_e = C_1 + C_2 x - (Q_0 L^2 / K_0) \exp(x/L)
\]

where \( C_1 \) and \( C_2 \) are integration constants. Applying the boundary conditions we find

\[
u_e(0) = C_1 - (Q_0 L^2 / K_0) = u_0 \quad \Rightarrow \quad C_1 = \frac{K_0 u_0 + Q_0 L^2}{K_0}
\]

\[
u_e(L) = C_1 + C_2 L - (Q_0 L^2 / K_0) e = u_L \quad \Rightarrow \quad C_2 = \frac{K_0 (u_L - u_0) + Q_0 L^2 (e - 1)}{K_0 L}
\]

where \( e = \exp(1) \). Thus

\[
\begin{array}{l}
u_e = \frac{K_0 u_0 + Q_0 L^2}{K_0} + \frac{K_0 (u_L - u_0) + Q_0 L^2 (e - 1) x}{K_0} - \frac{Q_0 L^2 \exp(x/L)}{K_0 L}
\end{array}
\]

- The flux \( \phi(x) \) is given by

\[
\begin{aligned}
\phi &= -K_0 u_e' \\
&= -K_0 [C_2 - (Q_0 L / K_0) \exp(x/L)] \\
&= Q_0 L \exp(x/L) - \frac{K_0 (u_L - u_0) + Q_0 L^2 (e - 1)}{L}
\end{aligned}
\]
where we have used our expression for $C_2$ found above. Evaluating the flux at $x = 0$ and $x = L$ we find

\[
\begin{align*}
\phi_0 &= \phi(x = 0) = -\frac{K_0(u_L - u_0) - Q_0L^2(e - 2)}{L} \\
\phi_L &= \phi(x = L) = -\frac{K_0(u_L - u_0) + Q_0L^2}{L}
\end{align*}
\]

Note that the net heat flux out of the rod $\phi_L - \phi_0$ is given by

\[
\phi_L - \phi_0 = Q_0L(e - 1)
\]

We will use this result later

• The thermal energy generated in the rod is given by $\int_0^L Q(x)A\,dx$. Using our expression for the source term we find

\[
\int_0^L Q(x)A\,dx = Q_0A \int_0^L \exp(x/L)\,dx = Q_0AL \exp(x/L)|_0^L = Q_0AL(e - 1)
\]

• Integrating the 1D Heat Equation within our rod we obtain the Heat Equation in integral form

\[
\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q
\]

\[
\rho c \frac{d}{dt} \int_0^L uA\,dx = \left[ K_0 A \frac{\partial u}{\partial x} \right]_0^L + \int_0^L Q(x)A\,dx
\]

In steady state (equilibrium) the time derivative vanishes leaving

\[
\int_0^L Q(x)A\,dx = A(\phi_L - \phi_0)
\]

In words, this states that in equilibrium the net thermal energy generated in the rod is equal to the net flux out of the rod. Looking at our results above for $\int_0^L Q(x)A\,dx$, $\phi_L$, and $\phi_0$, we see this condition is satisfied

**Problem 2:**

In the absence of internal heat sources, heat conduction in a rod of length $L$ with uniform density $\rho$, specific heat $c$, and thermal conductivity $K_o$ is determined by integration of

\[
\rho c \frac{\partial u}{\partial t} = K_o \frac{\partial^2 u}{\partial x^2}
\]

with initial condition $u(x, 0) = u_i(x)$ and appropriate boundary conditions at both ends. Investigate the existence of equilibrium temperature distributions $u_e(x)$ for $t \to \infty$ in the following cases:

• $u(0, t) = u_0, -K_o \frac{\partial u}{\partial x}(L, t) = \phi_L$
• $u(0, t) = u_0, -K_o \frac{\partial u}{\partial x}(L, t) = H[u(L, t) - u_B]$
• $-K_o \frac{\partial u}{\partial x}(0, t) = \phi_1, -K_o \frac{\partial u}{\partial x}(L, t) = \phi_1$

where $u_0$, $\phi_L$, $H$, $u_B$, and $\phi_1$ are given positive constants.
Solution 2:

In equilibrium the problem reduces to $\frac{d^2u_e}{dx^2} = 0$. Integrating we find the temperature satisfies a linear distribution given by $u_e = C_1 + C_2x$. We will solve for the integration constants $C_1$ and $C_2$ based off the various boundary conditions provided:

- $u(0, t) = u_0$, $-K_o \frac{\partial u}{\partial x}(L, t) = \phi_L$
  
  Applying the boundary conditions we find
  
  \[
  u_e(0) = C_1 = u_0 \quad \Rightarrow \quad C_1 = u_0 \\
  -K_0 u'_e(L) = -K_0 C_2 = \phi_L \quad \Rightarrow \quad C_2 = -\frac{\phi_L}{K_0}
  \]

  Thus we find
  
  \[
  u_e = u_0 - \frac{\phi_L x}{K_0}
  \]

- $u(0, t) = u_0$, $-K_o \frac{\partial u}{\partial x}(L, t) = H[u(L, t) - u_B]$
  
  Similarly we find $C_1 = u_0$. Applying the second boundary condition we find
  
  \[-K_0 C_2 = H[C_1 + C_2L - u_B] \]

  Solving for $C_2$ we obtain
  
  \[
  C_2 = \frac{H(u_B - u_0)}{K_0 + HL}
  \]

  Thus we find
  
  \[
  u_e = u_0 + \frac{H(u_B - u_0)}{K_0 + HL} x
  \]

- $-K_o \frac{\partial u}{\partial x}(0, t) = \phi_1$, $-K_o \frac{\partial u}{\partial x}(L, t) = \phi_1$
  
  Applying the boundary conditions we find
  
  \[-K_0 u'_e(0) = -K_0 C_2 = \phi_1 \quad \Rightarrow \quad C_2 = -\frac{\phi_1}{K_0} \\
  -K_0 u'_e(L) = -K_0 C_2 = \phi_1 \quad \Rightarrow \quad C_2 = -\frac{\phi_1}{K_0}
  \]

  Thus both boundary conditions are satisfied with $C_2 = -\phi_1/K_0$. In order to find $C_1$ we use the integral form of the heat equation along with our boundary conditions, namely

  \[
  \rho c \frac{d}{dt} \int_0^L u dx = [K_o \frac{\partial u}{\partial x}]_0^L \\
  = \phi_1 - \phi_1 \\
  = 0
  \]

  Therefore $\int_0^L u dx = \text{constant}$. Applying this result we find

  \[
  \int_0^L u_i(x) dx = \int_0^L C_1 + C_2 x \ dx
  \]
where \( u_i(x) \) is the initial temperature distribution throughout the rod. Solving for \( C_1 \) gives

\[
C_1 = \frac{\phi_1 L}{2K_0} + \frac{1}{L} \int_0^L u_i(x) \, dx
\]

Thus we find

\[
u_e = \frac{1}{L} \int_0^L u_i(x) \, dx + \frac{\phi_1 L}{2K_0} - \frac{\phi_1 x}{K_0}
\]

Problem 3:

Obtain the equilibrium temperature distribution \( u_e(x) \) corresponding to a bar of length \( L \) with constant thermal conductivity \( K_0 \) subject to a volumetric cooling rate \( Q = -C^2(K_0/L^2)u \) proportional to the temperature \( u \), with \( C \) being a positive dimensionless constant. Assume that both ends are kept at the same constant temperature \( u(0, t) = u(L, t) = u_0 \). Evaluate the heat fluxes at both ends and verify that \( \int_0^L Q(x) \, dx = \phi_L - \phi_0 \).

Solution 3:

We begin with

\[
\rho c \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q
\]

In steady state the problem reduces to solving

\[
\begin{align*}
u_e'' - \left( \frac{C^2}{L^2} \right) u_e &= 0 \\
u_e(0) = u_e(L) &= u_0
\end{align*}
\]

We recognize this as a constant coefficient ODE with solutions \( u_e \sim e^{rx} \) Inserting into our equation we find \( r^2 - C^2/L^2 = 0 \) with solution \( r = \pm C/L \). Hence

\[
u_e = C_1 e^{Cx/L} + C_2 e^{-Cx/L} = D_1 \cosh(Cx/L) + D_2 \sinh(Cx/L)
\]

where the we have introduced hyperbolic functions to make solving for the integration constants \( D_1 \) and \( D_2 \) easier. Applying the boundary conditions, and noting that \( \cosh(0) = 1 \) and \( \sinh(0) = 0 \) we find

\[
\begin{align*}
u_e(0) = D_1 + 0 &= u_0 \quad \Rightarrow \quad D_1 = u_0 \\
u_e(L) = u_0 \cosh(C) + D_2 \sinh(C) &= u_0 \quad \Rightarrow \quad D_2 = \frac{u_0[1 - \cosh(C)]}{\sinh(C)}
\end{align*}
\]

Thus

\[
u_e = u_0 \cosh(Cx/L) + \frac{u_0[1 - \cosh(C)]}{\sinh(C)} \sinh(Cx/L)
\]
The flux is given by $\phi(x) = -K_0u_e'$. Using our expression for $u_e$ above we find

$$\phi(x) = -K_0 \left[ \frac{u_0 C}{L} \sinh(Cx/L) + \frac{Cu_0[1 - \cosh(C)]}{L \sinh(C)} \cosh(Cx/L) \right]$$

$$\phi_0 = \phi(0) = -K_0 \frac{Cu_0[1 - \cosh(C)]}{L \sinh(C)}$$

$$\phi_L = \phi(L) = -K_0 \frac{u_0 C}{L} \sinh(C) + \frac{Cu_0[1 - \cosh(C)]}{L \sinh(C)} \cosh(C)$$

Thus the net flux out of the rod $\phi_L - \phi_0$ is given by

$$\phi_L - \phi_0 = -CK_0u_0 \frac{L}{L} \left[ \sinh(c) - \frac{(1 - \cosh(C))^2}{\sinh(C)} \right]$$

The net heat generated within the rod is given by

$$\int_0^L Q(x) \, dx = \frac{-C^2 K_0}{L^2} \int_0^L u_e \, dx$$

$$= \frac{-C^2 K_0}{L^2} \int_0^L \left[ u_0 \cosh(Cx/L) + \frac{u_0[1 - \cosh(C)]}{\sinh(C)} \sinh(Cx/L) \right] \, dx$$

$$= \frac{-C^2 K_0}{L^2} \left[ u_0 \frac{L}{C} \cosh(Cx/L) + \frac{u_0 L [1 - \cosh(C)]}{C \sinh(C)} \cosh(Cx/L) \right]_0^L$$

$$= \frac{-CK_0u_0}{L} \left[ \sinh(c) - \frac{(1 - \cosh(C))^2}{\sinh(C)} \right]$$

Hence as expected, we have shown $\int_0^L Q(x) \, dx = \phi_L - \phi_0$. 