Problem 1:
Consider the \textit{homogeneous} heat-conduction problem
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}; \quad u(x,0) = u_i(x), \quad u(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0 \]
describing the temporal evolution of the temperature \( u(x,t) \) along a constant-thermal-diffusivity rod of length \( L \) whose left end has a zero temperature and whose right end is insulated.

1. Determine the equilibrium temperature distribution \( u_e(x) \) found as \( t \to \infty \).
2. Introduce a product function of the form \( u = F(x)G(t) \). Anticipate the possible values of the separation constant \( \lambda \) based on the equilibrium distribution \( u_e(t) \).
3. Obtain the form of the functions \( F(x) \) and \( G(t) \). In solving for \( F(x) \) consider separately the cases \( \lambda > 0, \lambda = 0, \) and \( \lambda < 0, \) showing that only solutions with \( \lambda > 0 \) are possible.
4. Using the principle of superposition express \( u(x,t) \) as an infinite sum of product functions. Use the initial condition \( u(x,0) = u_i(x) \) to find expressions for the constant coefficients \( B_n \) multiplying each one of the terms, indicating all steps involved in the computation.
5. Obtain explicit expressions for the different coefficients \( B_n \) in the following cases:
   - \( u_i = 6 \sin \left( \frac{3\pi x}{2L} \right) + 2 \sin \left( \frac{5\pi x}{2L} \right) \)
   - \( u_i = \cos \left( \frac{3\pi x}{2L} \right) \sin \left( \frac{3\pi x}{2L} \right) \)
   - \( u_i = 1 \)

Solution 1:
1. Inserting \( u_e = u_e(x) \) into the heat equation we obtain
\[ u_e'' = 0; \quad u_e(0) = 0; \quad u_e'(L) = 0 \]
We integrate this twice to find the equation for a straight line. The boundary conditions imply that the straight line goes through zero at \( x = 0 \) and has zero slope. Hence
\[ u_e = 0 \]
2. Since \( u_e \neq \text{constant} \) we anticipate that \( \lambda = 0 \) will \textbf{not} be an eigenvalue of our problem. We will see this in more detail below.
3. Introducing a separable form of the solution \( u = F(x)G(t) \) into the heat equation yields two homogenous constant coefficient ODEs, namely
\[ G' + k\lambda G = 0 \]
\[ F'' + \lambda F = 0; \quad F(0) = 0; \quad F'(L) = 0 \]
Integrating the equation for $G$ we find $G = Ce^{-k\lambda t}$. Hence for the solution to remain bounded as $t \to \infty$ we only consider $\lambda \geq 0$. Now considering $\lambda = 0$ and $\lambda > 0$ separately we integrate to find

$$
\lambda = 0 \quad \Rightarrow \quad F = Ax + D
$$

$$
\lambda > 0 \quad \Rightarrow \quad F = A \cos(\sqrt{\lambda}x) + D \sin(\sqrt{\lambda}x)
$$

For $\lambda = 0$, applying the boundary conditions gives $F = 0$ which gives $u = 0$ (trivial solution). Hence we conclude that $\lambda > 0$. Applying the boundary conditions for $\lambda > 0$ we find

$$
F(0) = 0 = A
$$

$$
F'(L) = 0 = \sqrt{\lambda}D \cos(\sqrt{\lambda}L)
$$

Thus either $D = 0 \Rightarrow F = 0 \Rightarrow u = 0$ (trivial solution) or $\cos(\sqrt{\lambda}L) = 0$. Noting that $\cos(\phi_n) = 0$ when $\phi_n = (n - 1/2)\pi = \pi/2, 3\pi/2, 5\pi/2, \ldots$ $n = 1, 2, 3, \ldots$ we find our eigenvalues are given by

$$
\lambda_n = \left[\frac{(n - 1/2)\pi}{L}\right]^2 \quad n = 1, 2, 3, \ldots
$$

4. Hence all solutions of the form $F_n = D_n \sin(\sqrt{\lambda_n}x)$ and $G_n = C_n e^{-k\lambda_n t}$ $n = 1, 2, 3, \ldots$ satisfy our ODEs (3) along with appropriate boundary conditions. Moreover, for each $n$, the product $u_n = F_n G_n$ is a solution to our PDE along with appropriate boundary conditions. Since our equation is linear we may use superposition to express the general solution as $u(x, t) = \sum_{n=1}^{\infty} u_n$. Thus we find

$$
u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k\lambda_n t} \sin(\sqrt{\lambda_n}x)
$$

where $B_n = C_n D_n$. Now using our initial condition to find $B_n$ we find

$$
u(x, 0) = u_i(x) = \sum_{n=1}^{\infty} B_n \sin(\sqrt{\lambda_n}x)
$$

Multiplying by $\sin(\sqrt{\lambda_m}x)$ and integrating over the domain gives

$$
\int_0^L u_i(x) \sin(\sqrt{\lambda_m}x)dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin(\sqrt{\lambda_n}x) \sin(\sqrt{\lambda_m}x)dx = B_m \frac{2}{L}
$$

where we have made use of the orthogonality relation

$$
\int_0^L \sin(\sqrt{\lambda_n}x) \sin(\sqrt{\lambda_m}x)dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{L}{2}, & \text{if } n = m \neq 0 \end{cases}
$$

Thus

$$
B_n = \frac{2}{L} \int_0^L u_i(x) \sin(\sqrt{\lambda_n}x)dx
$$

The final solution is given by Equations (6), (7) and (10)
5. To find explicit expressions for $B_n$ we will use inspection of Equation (8) for simple $u_i(x)$ or apply Equation (10) directly

- $u_i = 6 \sin \left(\frac{3\pi x}{2L}\right) + 2 \sin \left(\frac{5\pi x}{2L}\right)$

  Applying Equation (8) we find $u_i = 6 \sin \left(\frac{3\pi x}{2L}\right) + 2 \sin \left(\frac{5\pi x}{2L}\right) = \sum_{n=1}^{\infty} B_n \sin(\sqrt{\lambda_n} x)$. Thus by inspection we find $B_2 = 6$ and $B_3 = 2$. All other $B_n = 0$

- $u_i = \cos \left(\frac{3\pi x}{2L}\right) \sin \left(\frac{\pi x}{L}\right)$

  We may use the fact that $\sin(a) \cos(b) = \frac{1}{2}[\sin(a + b) + \sin(a - b)]$ to rewrite $u_i(x) = \frac{1}{2} \sin(\frac{3\pi x}{2L})$ Applying Equation (8) we find $u_i(x) = \frac{1}{2} \sin(\frac{3\pi x}{2L}) = \sum_{n=1}^{\infty} B_n \sin(\sqrt{\lambda_n} x)$. Thus by inspection we find $B_2 = \frac{1}{2}$. All other $B_n = 0$

- $u_i = 1$

  We apply Equation (10) with $u_i = 1$. This gives

$$B_n = \frac{2}{L} \int_0^L (1) \sin(\sqrt{\lambda_n} x) dx$$

$$= -\frac{2}{L} \left[ \cos(\sqrt{\lambda_n} x) \right]_0^L$$

$$= -\frac{2}{L \sqrt{\lambda_n}} \left[ \cos(\sqrt{\lambda_n} L) - 1 \right]$$

$$= \frac{2}{(n - 1/2)\pi}$$

where we have used the definition of $\lambda_n$ and the fact that $\cos(\sqrt{\lambda_n} L) = 0$

Problem 2:

Consider the non-homogeneous heat-conduction problem

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}; \quad u(x, 0) = u_i(x), \quad -K_0 \frac{\partial u}{\partial x}(0, t) = \phi_0, \quad u(L, t) = u_L$$

describing the temporal evolution of the temperature in a constant-thermal-diffusivity rod of length $L$ with a prescribed heat flux at the left end and a prescribed temperature at the right end.

1. Determine the equilibrium temperature distribution $u_e(t)$ found as $t \to \infty$.

2. Introduce the temperature departure from equilibrium $v(x, t) = u(x, t) - u_e(x)$. Write the equation with initial and boundary conditions satisfied by $v(x, t)$.

3. Use separation of variables to determine the function $v(x, t)$ as an infinite sum of product functions, giving expressions for the different constant coefficients $A_n$ multiplying each one of the terms. Indicate clearly all steps involved in the computation.

4. Obtain the solution for $u(x, t)$ in the particular case $u_i = 0$. 

Solution 2:

1. Inserting $u_e = u_e(x)$ into the heat equation we obtain

$$u_e'' = 0; \quad u_e'(0) = -\phi_0/K_0; \quad u_e(L) = u_L$$

Integrating we find the equation of a straight line of slope $-\phi_0/K_0$ through $u_e = u_L$ at $x = L$. The corresponding solution is

$$u_e = \frac{\phi_0}{K_0}(L - x) + u_L$$

2. Letting $u(x, t) = v(x, t) + u_e(x)$ and inserting into our equation yields

$$\frac{\partial[v(x, t) + u_e(x)]}{\partial t} = k \frac{\partial^2[v(x, t) + u_e(x)]}{\partial x^2}$$

since $\frac{\partial u_e}{\partial t} = 0$ and $\frac{\partial^2 u_e}{\partial x^2} = u_e'' = 0$

Our initial condition becomes

$$v(x, t) = u(x, t) - u_e(x)$$

$$v(x, 0) = u(x, 0) - u_e(x)$$

$$\Rightarrow \quad v_i(x) = u_i(x) - u_e(x)$$

The first boundary condition becomes

$$\frac{\partial[v(x, t) + u_e(x)]}{\partial x}(0, t) = -\frac{\phi_0}{K_0}$$

$$\frac{\partial v}{\partial x}(0, t) - \frac{\phi_0}{K_0} = -\frac{\phi_0}{K_0}$$

$$\Rightarrow \quad \frac{\partial v}{\partial x}(0, t) = 0$$

since $\frac{\partial u_e}{\partial x} = u_e' = -\phi_0/K_0$

Similarly the second boundary condition becomes

$$v(L, t) = 0$$

since $u_e(L) = u_L$. In summary, $v(x, t)$ must satisfy the \textbf{homogeneous} problem

$$\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}; \quad v(x, 0) = v_i(x) = u_i(x) - u_e(x), \quad \frac{\partial v}{\partial x}(0, t) = 0, \quad v(L, t) = 0$$

3. Now solving for $v(x, t)$ we may propose a solution of the form $v = F(x)G(t)$ and proceed with separation of variables as usual. However, it is easier to note the problem for $v(x, t)$, namely Equation (18), is exactly the same as Problem 1 except the zero temperature and zero flux boundary conditions have switched ends. Let's exploit this fact by introducing the change of variables $y = L - x$. This implies that when $x = 0$, $y = L$ and when $x = L$, $y = 0$. Moreover

$$\frac{\partial}{\partial x} = \frac{dy}{dx} \frac{\partial}{\partial y} = -\frac{\partial}{\partial y}$$
and similarly
\[
\frac{\partial^2}{\partial x^2} = \frac{\partial y}{\partial x} \frac{\partial}{\partial y} \left( -\frac{\partial}{\partial y} \right) = \frac{\partial^2}{\partial y^2}
\]
Thus Equation (18) reduces to
\[
\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial y^2} ; \quad v(0, t) = 0, \quad \frac{\partial u}{\partial y}(L, t) = 0
\]
which has been solved in Problem 1! Using the solution in Equation (7) with \( x \) replaced by \( y \) we find
\[
v(y, t) = \sum_{n=1}^{\infty} B_n e^{-k\lambda_n t} \sin(\sqrt{\lambda_n} y)
\]
\[
\lambda_n = \left[ \left( n - \frac{1}{2} \right) \frac{\pi}{L} \right]^2 \quad n = 1, 2, 3, \ldots
\]
Now inserting \( y = L - x \) into Equation (19) we find
\[
v(x, t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda_n t} \cos(\sqrt{\lambda_n} x)
\]
\[
\lambda_n = \left[ \left( n - \frac{1}{2} \right) \frac{\pi}{L} \right]^2 \quad n = 1, 2, 3, \ldots
\]
where we have used \( \sin(\sqrt{\lambda_n} (L - x)) = (-1)^{n-1} \cos(\sqrt{\lambda_n} x) \) and \( A_n = (-1)^{n-1} B_n \) Now using our initial condition to find \( A_n \) we find
\[
v(x, 0) = v_i(x) = \sum_{n=1}^{\infty} A_n \cos(\sqrt{\lambda_n} x)
\]
Multiplying by \( \cos(\sqrt{\lambda_m} x) \) and integrating over the domain gives
\[
\int_0^L v_i(x) \cos(\sqrt{\lambda_m} x) \, dx = \sum_{n=1}^{\infty} A_n \int_0^L \cos(\sqrt{\lambda_n} x) \cos(\sqrt{\lambda_m} x) \, dx
\]
\[
= A_m \frac{2}{L}
\]
where we have made use of the orthogonality relation
\[
\int_0^L \cos(\sqrt{\lambda_n} x) \cos(\sqrt{\lambda_m} x) \, dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{L}{2}, & \text{if } n = m \neq 0 \end{cases}
\]
Thus
\[
A_n = \frac{2}{L} \int_0^L v_i(x) \cos(\sqrt{\lambda_n} x) \, dx
\]
4. For the specific case that \( u_i = 0 \) we have
\[
v_i(x) = -u_e = -ax - b
\]
where we have introduced \( a = -\phi_0/K_0 \) and \( b = \phi_0 L/K_0 + u_L \) to ease notation. Applying Equation (25) directly we find

\[
A_n = \frac{-2}{L} \int_0^L (ax + b) \cos(\sqrt{\lambda_n}x) \, dx
\]

\[
= \frac{-2}{L} \left( a \int_0^L x \cos(\sqrt{\lambda_n}x) \, dx + b \int_0^L \cos(\sqrt{\lambda_n}x) \, dx \right)
\]

\[
= \frac{-2}{L} \left( a \left[ \frac{\sqrt{\lambda_n}x \sin(\sqrt{\lambda_n}x)}{\lambda_n} + \frac{\cos(\sqrt{\lambda_n}x)}{\lambda_n} \right]_0^L + b \sqrt{\lambda_n} \sin(\sqrt{\lambda_n}x) \right)
\]

\[
= \frac{-2}{L \sqrt{\lambda_n}} \left( a((-1)^{n-1}L - 1) + b((-1)^{n-1}) \right)
\]

Thus

\[
A_n = \frac{-2}{(n - 1/2)\pi} \left( a((-1)^{n-1}L - 1) + b((-1)^{n-1}) \right)
\]

(26)

The final solution is given by \( u(x, t) = v(x, t) + u_e(x) \) where \( v(x, t) \) is completely described by Equations (21),(22) and (26)
Problem 3:

Consider the **non-homogeneous** heat-conduction problem

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + \frac{Q(x)}{\rho c} \quad ; \quad u(x,0) = u_i(x), \quad u(0,t) = u_0, \quad u(L,t) = u_L
\]

describing the temporal evolution of the temperature in a constant-thermal-diffusivity rod of length \(L\) with prescribed end temperatures subject to a volumetric-heat-source distribution \(Q(x)\). For the analysis below consider in particular the volumetric heating rate \(Q(x)/(\rho c) = 12A(x/L)^2\).

1. Determine the equilibrium temperature distribution \(u_e(t)\) found as \(t \to \infty\).

2. Write the equation with initial and boundary conditions satisfied by the temperature departure from equilibrium \(v(x,t) = u(x,t) - u_e(x)\).

3. Use separation of variables to determine the function \(v(x,t)\) as an infinite sum of product functions, giving expressions for the different constant coefficients \(B_n\) multiplying each one of the terms. Indicate clearly all steps involved in the computation.

Solution 3:

1. Inserting \(u_e = u_e(x)\) into our equation we find

\[
0 = u_e'' + (12A/kL^2)x^2 = 0; \quad u_e(0) = u_0, \quad u_e(L) = u_L
\]

Integrating twice we find

\[
u_e = -(A/kL^2)x^4 + Cx + B
\]

Applying our boundary conditions we obtain

\[
u_e(0) = u_0 = B \quad \Rightarrow \quad B = u_0
\]
\[
u_e(L) = u_L = -AL^2/k + CL + u_0 \quad \Rightarrow \quad C = \frac{u_L - u_0}{L} + \frac{AL}{k}
\]

Hence

\[
u_e = -(A/kL^2)x^4 + \left[\frac{u_L - u_0}{L} + \frac{AL}{k}\right]x + u_0
\]  

(27)

2. Now with \(u(x,t) = v(x,t) + u_e(x)\) our PDE becomes

\[
\frac{\partial v}{\partial t} - k \frac{\partial^2 v}{\partial x^2} = \left[\frac{\partial u_e}{\partial t} - k \frac{\partial^2 u_e}{\partial x^2} - \frac{Q(x)}{\rho c}\right] = 0
\]

(28)

where the last equality holds since the term in brackets is zero (we used this to solve for \(u_e\) above in part 1)

Our boundary and initial conditions become

\[
u(0,t) = u_0 = u_e(0) + v(0,t) \quad \Rightarrow \quad v(0,t) = 0
\]

(29)
\[
u(L,t) = u_L = u_e(L) + v(L,t) \quad \Rightarrow \quad v(L,t) = 0
\]

(30)
\[
u(x,0) = u_i(x) = u_e(x) + v(x,0) \quad \Rightarrow \quad v(x,0) = u_i(x) - u_e(x)
\]

(31)

In summary, \(v(x,t)\) must satisfy the **homogeneous** problem

\[
\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}; \quad v(x,0) = v_i(x) = u_i(x) - u_e(x), \quad v(0,t) = v(L,t) = 0
\]

(32)
3. Equation (32) is the heat equation with zero temperature at both ends. This problem has been solved in both lecture and in Section 2.3 of Haberman. The familiar solution is given by

\[ v(x, t) = \sum_{n=1}^{\infty} B_n e^{-k(n\pi/L)^2 t} \sin(n\pi x/L) \]  

(33)

\[ B_n = \frac{2}{L} \int_0^L v_i(x) \sin(n\pi x/L) \, dx \]  

(34)

where \( v_i(x) = u_i(x) - u_e(x) \)